Quantum Control of Nonlinear Dynamics in Confined Fluids

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Investigating nonlinear fluid dynamics remains a challenge across physics from nanofluidics and biophysics to astrophysics. Here we introduce a quantum/classical theoretical approach that takes into account both quantum correlations and classical behaviour within a 2D fluid that is confined in a 3 µm side square. We employ a modified Gross-Pitaevskii equation, encompassing many-body interactions and confinement. This system reveals complex fluid dynamics characterised by dissipative solitons; a significant outcome is an asymptotic function that describes the soliton behaviour. The solitons exhibit intriguing geometrical and temporal transformations, guided by subtle phase gradients. We trace the soliton evolution from 1 to 83 ns, revealing the emergence of geometric oscillations in amplitude and phase angles. Under these phase gradients, solitons transition to states with reduced amplitude and expanded spatial profiles. These results show that geometric solitons can emerge from a quantum noisy environment, and lead us to propose an interesting possibility: it is feasible to control and manipulate nonlinear dynamics in systems with finite-range interactions and confinement using quantum control. By bridging quantum and classical dynamics, this study links various scientific disciplines, including non-equilibrium phases of condensed matter, unconventional/quantum computing and advanced control of nanofluidics. From a more fundamental perspective, this possibility of quantum control of classical behaviour advances our understanding of physics within multidimensional Hilbert spaces.

I. INTRODUCTION

The role of complexity of nonlinear dynamics within confined fluidic systems expands across disciplines, offering an intriguing intersection of physics, mathematics, and real-world applications [1–5]. In such systems, the behaviour of molecules and waves defies conventional expectations, often revealing hidden complexities that challenge our understanding [6–8]. Within these confined domains, solitons, those elusive solitary waves characterised by their ability to maintain their shape as they propagate, have emerged as prominent actors [9–15]. Their extraordinary stability and widespread occurrence, from the control of mode-locked lasers to the propagation of action potentials in cell membranes and neurons [16], makes them the subject of widespread investigation across disciplines. In nanofluidics, where the behaviour of fluids at the nanoscale challenges classical analysis and interpretation, solitons have been hypothesized to play a role in fluid transport and actuation [17]. In this paper we will explore the role of quantum domains within confined fluids; for the development of the theory we find inspiration not only in the rich history of nonlinear physics but also in the groundbreaking work on the mathematical modelling of astrophysical singularities [18]. The complexities and paradoxes that nonlinear dynamics often present are reminiscent of the challenges Chandrasekhar grappled with in his work [19], offering us a unique opportunity to bridge seemingly disparate disciplines.

From a condensed matter physics perspective, our work studies unexplored aspects on behaviour in the quantum/classical transition in non-equilibrium systems. In the quantum/classical regime characteristic length scales are large enough for quantum states to lose their phase memory and non-unitary processes to affect the evolution of the quantum states, giving rise to unique phenomena which are the focus of much current attention. These phenomena are forbidden in the quantum world described by Schrödinger's equation and are also disallowed in the classical regime [20]. The extension of many-body quantum dynamics to the non-unitary domain has led to a series of exciting developments, including new out-of-equilibrium entanglement phases and phase transitions and emergent patterns of quantum information in space–time [21]. It has been shown that in these systems quantum noise can be used as an independent probe of the phases at accessible system sizes [22]. In photonic systems, Bose–Einstein condensation has revealed a rich phenomenology related to spontaneous coherence generation in driven-dissipative spatially extended systems and is providing a new platform for the study of non-equilibrium phase transitions and critical behaviours which lead to rich mean-field dynamics, such as condensation in excited states, topological lasing, outward flows in localised condensates, spiralling condensate phases around quantised vortices, diffusive Goldstone modes and generalised Landau criteria for superfluidity [23].

The solution of nonlinear partial differential (NLPD) equations within confined spaces, particularly in the con-

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text of fluid dynamics, poses substantial computational challenges. These equations often feature hyperbolic and exponential terms that introduce complexities, potentially leading to inaccuracies and a weakening of the nonlinear component. Consequently, it becomes necessary to seek solution forms that offer enhanced stability, mitigating dispersion effects within the system [24]. Although various methodologies have been employed to tackle NLPD equations [25–27], there remains a notable gap in understanding the dependence of solutions on classical parameters. Furthermore, the visualisation of these solutions is frequently overlooked. One prominent domain where such nonlinear dynamics arise is the study of confined quantum systems, exemplified by trapped Bose-Einstein condensates. These systems, described using the Gross-Pitaevskii (GP) equation, exhibit many-body interactions in nanometric spaces. However, achieving such conditions often necessitates external boundary conditions, such as intense magnetic fields or extremely low temperatures [28]. The presence of multiple species with anisotropic properties in these short-range interactive spaces further amplifies the complexity of their dynamics.

The GP equation, which can be viewed as a specialised form of the nonlinear Schrödinger equation, accommodates an additional nonlinearity term to account for interspecies interactions:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + gN|\psi|^2\right)\psi \qquad (1)$$

Here, V_{ext} represents the potential of the confined system, and $g = 4\pi\hbar^2(a/m)$, where a is the scattering length. Additionally, in this work, we incorporate an extra term related to spins to address the behaviour of vortices and predict the system's nonequilibrium dynamics more precisely and accurately [29, 30]. Notably, in systems featuring oscillator potentials, exact solutions for the Gross-Pitaevskii equation remain elusive [31]. Instead, solutions dynamically evolve with changes in time-dependent parameters. In this paper, we embark on a comprehensive exploration of these challenges and opportunities, presenting three distinct solutions to the GP equation within the confines of our research domain.

II. THEORETICAL FOUNDATION

The exact solution to a Nonlinear Partial Differential Equation (NLPDE) is often constrained by specific operating conditions, which are intricately tied to the theoretical models governing the system and dependent on certain parameters [32]. Attempting to encapsulate the entire dynamics of such systems under these constraints is inherently complex. Dissipative solitons, on the other hand, emerge as a viable solution within nonlinear systems, owing to the interplay between dissipative and dispersive coefficients. These solitons exhibit

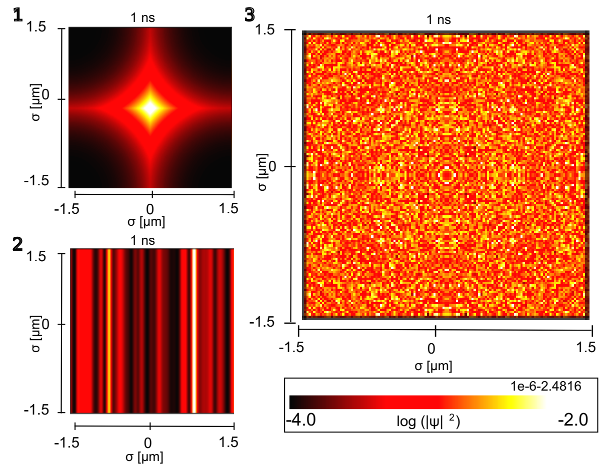

FIG. 1. (1) Evolution of the asymptotic curve in the numerical analysis of the 2D anisotropic Gross-Pitaevskii (GP) equation, independent of temperature, within a square domain of 3 μm in length. (2) Formation dynamics of observed solitons in the numerical analysis of the 2D anisotropic GP equation, accounting for temperature dependency, within a square domain of 3 μm in length. (3) Dynamics of observed condensates in the numerical analysis of the 2D anisotropic GP equation, considering temperature as a function of critical density, within a square domain of 3 μm in length.

remarkable stability and find utility in diverse applications, such as mode-locked lasers [33]. It is important to note that alterations in the aforementioned variables significantly influence the overall dynamics of a nonlinear system. In this discussion, we present a selection of exact solutions for nonlinear systems, drawing comparisons with Kudryashov's method for solving higher-order differential equations [34]. The influence of dispersive effects on the medium has been the subject of extensive research over the years. Lan et al. [35] have delved into the variational formalism for dispersive equations within nonlinear media, revealing solitonic behaviour. Additionally, Karpman et al. [36], have examined the stability of soliton waves using the Lyapunov approach, a methodology that we also verify in our research, as illustrated in Figure 1.1. In the context of classical physics, solving equations of motion provides a comprehensive understanding of a system's dynamic variables at any given time, enabling the depiction of its complete behaviour. However, in the context of quantum mechanics, the equation of motion transforms into the variation of expectation values over time for the state vector within the abstract Hilbert space. We introduce a linear operator, represented by the unitary operator $\hat{U}$, specifically tailored to this quantum system. This operator accommodates the system's anisotropy along the x and y axes while disregarding couplings along the z-axis. The wave function can be expressed as follows:

$$\psi(t) = \hat{U}(t, t_0)\psi(t_0) \qquad (2)$$

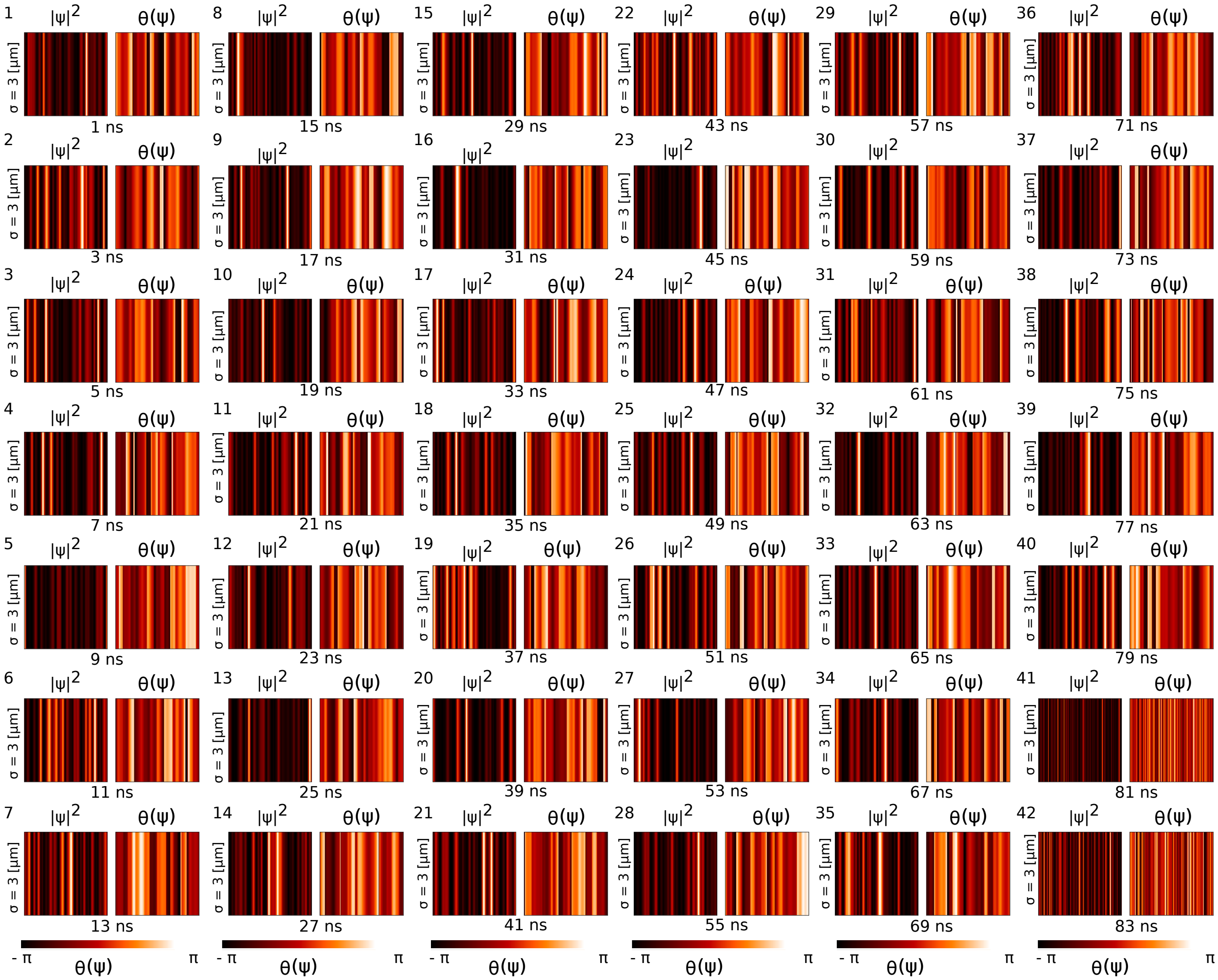

FIG. 2. Time evolution of soliton formation in a 2D anisotropic GP Equation (Hamiltonian in Equation 12) from 1 ns to 83 ns. Subfigures 1 to 42: Show the simultaneous plotting of the soliton's wave function amplitude ($|\psi|^2$) and phase angle ($\theta(\psi)$) with a 2 ns time step.

Here, $\hat{U}$ is unitary ($U^\dagger U = 1$, $U^\dagger = U^{-1}$), and it corresponds to the system. Ensuring the normalization condition of ψ ($\langle\psi(t)|\psi(t)\rangle = 1$) at the instants t and t_0 respectively, we have:

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle \tag{3}$$

$$\langle\psi(t)| = \langle\psi(t_0)|U^\dagger(t,t_0) \tag{4}$$

We then define a Hermitian operator $\hat{H}$ corresponding to the unitary operator $\hat{U}$ as:

$$\hat{U} = \exp(i\alpha\hat{H}) \tag{5}$$

Here, α represents the parameter of change throughout the simulation. In a prior study, the annihilation operator influenced the flow-switching mechanism of nanofluidic pores, as eigenstates may not be analytical signals within the interaction space [17]. To investigate the time evolution of such a system, we assume that the state vector ψ_H remains time-independent, while the spin operator $\hat{S}_H$ becomes time-dependent. Within Heisenberg's picture of the state vector, we have:

$$|\psi_H(t)\rangle = \hat{U}^{-1}(t,t_0)|\psi(t)\rangle \tag{6}$$

$$\hat{S}_H(t) = \hat{U}^{-1}(t,t_0)\hat{S}\hat{U}(t,t_0) \tag{7}$$

Now, assuming coherent states and a time-dependent spin operator governed by the dynamic variable θ, Equation 5 becomes:

$$U^{\dagger}(\theta)S_iU(\theta) = \exp[-i, S_{ix}\theta] \tag{8}$$

$$U^{\dagger}(\theta)S_iU(\theta) = S_{ix} \tag{9}$$

$$U^{\dagger}(\theta)S_{iy}U(\theta) = S_i\cos\theta - S_{iz}\sin\theta \tag{10}$$

$$U^{\dagger}(\theta)S_{iz}U(\theta) = S_{iy}\sin\theta + S_{iz}\cos\theta \tag{11}$$

Given that we do not consider z-axis coupling interactions, the differential transformation along the x-axis is zero. The transformation along the y-axis can be expressed as: $\frac{dS_{iy}}{d\theta}\Big|_{\theta=0} = -S_{iz}$. Subsequently, the Hamiltonian of a system incorporating the spin operator is given by:

$$H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + \frac{i}{2}\left(\frac{P}{1+\frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_i}{d\theta}\right]\psi \tag{12}$$

$$\frac{dS_i}{d\theta} = U^{\dagger}(\theta)S_{ix}U(\theta) + U^{\dagger}(\theta)S_{iy}U(\theta) + U^{\dagger}(\theta)S_{iz}U(\theta) \tag{13}$$

As we do not consider z-axis coupling interactions, the above expression simplifies to $\frac{dS_i}{d\theta} = \frac{d}{d\theta}(S_i\cos\theta)$ Thus, Equation 12 becomes:

$$H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + \frac{i}{2}\left(\frac{P}{1+\frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_i\cos\theta}{d\theta}\right]\psi \tag{14}$$

III. DYNAMICS IN A CONFINED DOMAIN

The time evolution of nonlinear dynamical systems often yields soliton solutions through numerical analysis. Some systems also exhibit vortices, spiral solutions, and asymptotic solutions. Figure 1.1 illustrates the temperature-independent solution of nonlinear dynamics within a confined space. In this simulation, we solve the Gross-Pitaevskii (GP) equation while considering the XY model within the Heisenberg picture. The solution assumes an asymptotic form at 1 ns, which eventually converges to a singular point by the end of 4 ns. Figure 1.2 presents the soliton solution within the same mathematical framework, albeit with potential expressed in inverse Fourier transform space. Figure 1.3 showcases the solution for the same Hamiltonian as before, but with additional parameters and operating conditions, including temperature and density of states, which are functionally dependent on geometry. This includes considering the relativistic wavelength and critical temperature as functions of the density of states.

A. NLGP Solution to Solitons in Confined 2D Space

The stochastic nonlinear Schrödinger equation [37], includes both the deterministic and stochastic components of the Hamiltonian. This Hamiltonian introduces quantum fluctuations (noise) in dispersive nonlinear systems, leading to the formation of quantum solitons. In such systems, nonlinearity effectively balances dispersion terms, resulting in the formation of nondispersive soliton waves [38]. We observe both bright and dark quantum solitons as solutions to the GP equation within the XY model. These observations occur over a confined square domain with a length of 3 μm. A bright soliton corresponds to a peak in the amplitude of the wavefunction, while a dark soliton exhibits a decrease in amplitude. The results are presented in Figure 2, covering a total period of 83 ns with a time step size of 2 ns.

In this simulation, the effective interaction between particles in the short-range interaction regime at the lowest energy range is denoted as $U = 4 \times \hbar^2 g/m$, where $U(\boldsymbol{r} - \boldsymbol{r_0})$ involves the position vectors $\boldsymbol{r}$ and $\boldsymbol{r_0}$ of two interacting particles. Additionally, the constants m (representing the unitary mass of the particle), P (saturation pumping strength of the particles) = 2, and system loss $q = 0.3$ are considered. The density is expressed as a function of the effective interaction between particles in a two-dimensional space:

$$n_0 = \rho \times \left(\frac{P}{q-1}\right) \tag{15}$$

Here, ρ denotes the particle density, and n_c is the critical density of the system. For a constant two-dimensional potential in Fourier-transformed space, the initial boundary conditions are $\psi = 1$, $\psi(t) = 1$, where $\psi(t)$ represents the wave function at time t. Several parameters vary in the simulation. The critical temperature (T_c) of the system, also a function of density, is given by: $T_c = 3.3\hbar^2\rho^{\frac{2}{3}}/mk_B$ The relativistic wavelength λ is defined as: $\lambda = \frac{\hbar}{\sqrt{(4\pi mk_BT_c)}}$ The relativistic energy E_n is expressed as: $E_n = 1.23/\lambda$ The chemical potential Q is defined by: $Q = \frac{n_0E_n - Tk_B}{n_0}$.

In Figure 2, the observed result of bright and dark quantum solitons along with their phase angle in two-dimensional space in the system under consideration is shown. These solitons, being prominent features of nonlinear systems, play a crucial role in understanding the

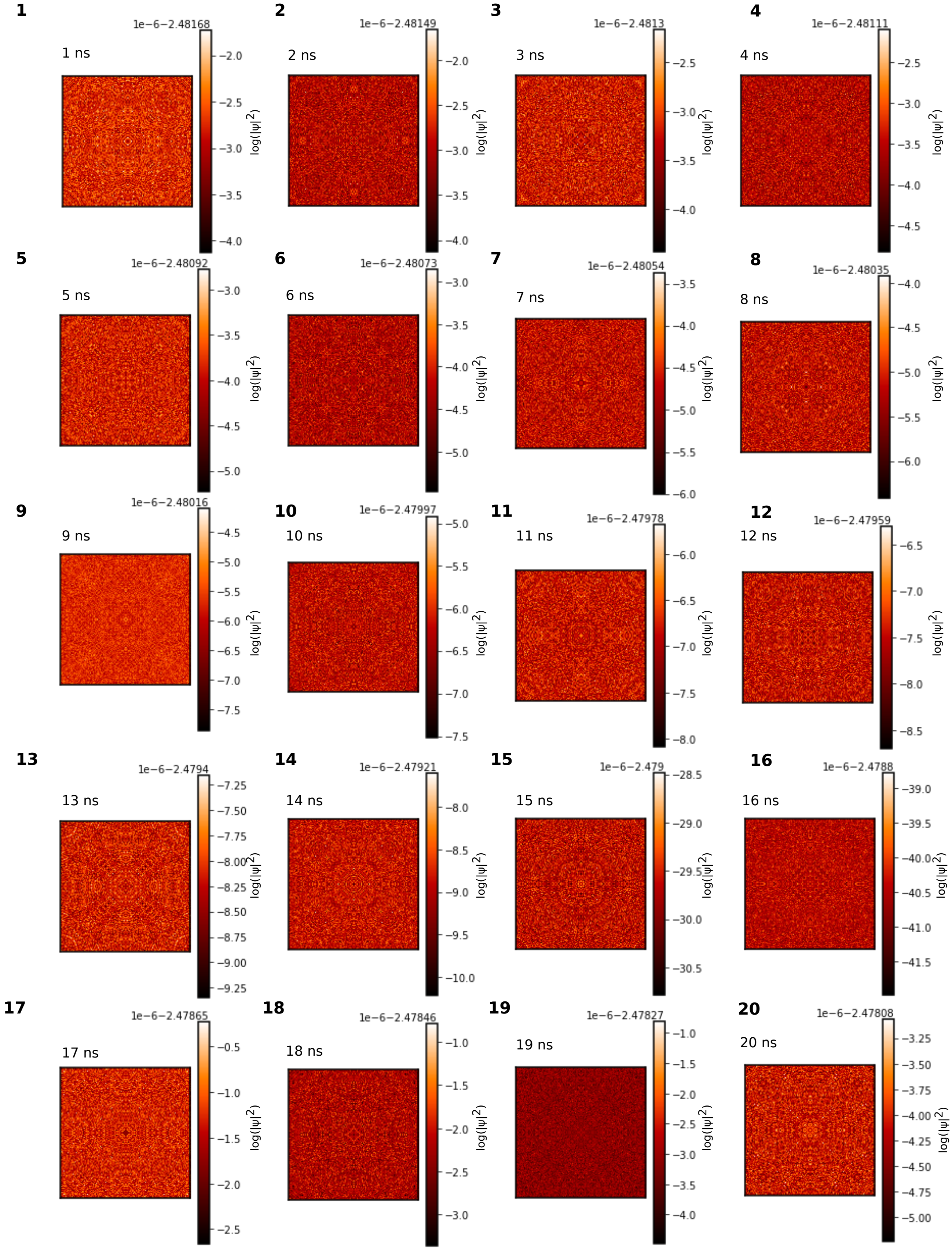

FIG. 3. 1 - 20: Growth dynamics and nucleation of BEC with time according to equation 14 – initial time frame = 1 ns with time steps of 1 ns and final time frame = 20 ns with time step = 1 ns.

system's dynamics. Bright solitons, characterised by peaks in the amplitude of the wavefunction, are observed in the figures as regions where the wavefunction amplitude remains relatively constant. Conversely, dark solitons manifest as areas where the amplitude of the wavefunction drops significantly. The presence and evolution of these solitons provide valuable insights into how the system responds to the interplay between nonlinearity and dispersion. As we progress through the figures, which span different time intervals, we witness a dynamic transformation in the solitons' patterns and interactions. At earlier time points, we observe isolated bright solitons with well-defined positions and amplitudes. As time advances, these solitons can merge, split, or even disappear, reflecting the intricate interplay between various factors influencing their behaviour. Additionally, we analyse the phase angles associated with these solitons. The phase angle of a soliton essentially describes the relative phase of its wavefunction - refers to how the solitons' phases change spatially. The phase angle provides critical information about the solitons' stability and their interactions with other solitons or perturbations in the system. In Figure 2.1, the bright soliton (constant amplitude) has the most stable modulation along 2 μm and relatively lower stability in the same 1 μm. In the case of its phase angle, these two amplitudes are anti-symmetric that is π and $-\pi$ respectively. At 3 ns in Figure 2.2, brighter solitons have appeared with prominent stability at 2 μm as in the previous case. At 5 ns in Figure 2.3, now the brightest soliton with at most stable amplitude has shifted to $\sigma = 1$ μm. At 7 ns in Figure 2.4, there are three soliton waves with similar amplitudes distributed randomly at $\sigma = 0.7$, 1, 2.9 respectively. At 9 ns in Figure 2.5, there are no bright solitons as in earlier cases. At 11 ns in Figure 2.6, the bright solitons are observed again and get prominent towards the centre region at 13 ns in Figure 2.7. At 15 ns in Figure 2.8 there is only a single prominent soliton wave at $\sigma = 1$ μm and direction of propagation is not clear. This single soliton wave gets split into 2 at 17 ns in Figure 2.9. They come further at proximity within 0.5 μm at 19 ns in Figure 2.10. Now the collision gets higher with time as there are no more stable solitons in the system at 21 ns in Figure 2.11. At 23 ns in Figure 2.12, wave are self strengthened to form 2 bright solitons at $\sigma = 1$ μm, 2 μm. At 25 ns in Figure 2.13, the soliton waves have moved apart to positions $\sigma = 0.9$ μm, 2.9 μm respectively. At 27 ns in Figure 2.14, the solitons are at proximity as in Figure 2.10. Here they are brighter, hence highly stable than earlier. At 29 ns in Figure 2.15, random distribution of solitons is observed with varying brightness. In Figure 2.16 - 2.22, the solitons do not follow a trend to consider their dynamics predictable. After a periodic interval, at 23 ns in Figure 2.23, only a single soliton wave with the highest stability is observed. In Figures 2.24 - 2.40, the positions of varying bright and dark solitons are random. In Figures 2.41 the collisions increase to a greater extent with dark soliton number exceeding the bright solitons. This trend increases further at 83 ns, where the solitons disappear and the amplitude of the wave function diminishes. After 83 ns, the amplitude of the wave functions goes to infinity and hence no solitons are observed after Figures 2.42. Now, the existence of these localised soliton waves alone does not guarantee the possibility of no other interaction in the system. Hence the coexistence of mixed interaction solutions along with the localized soliton waves [39] has to be explored further.

B. Evolution of NLGP Solution to Condensates in Confined 2D Space

We explore the dynamic growth of Bose-Einstein condensates (BEC), a unique quantum phenomenon, within a confined two-dimensional space. The mathematical framework for this exploration involves solving the Gross-Pitaevskii (GP) equation using the XY-model in the Heisenberg picture, employing the numerical integration technique known as the Runge-Kutta method. The visual representation of these dynamics is depicted in Figure 3. Figure 3 provides a chronological overview of the nucleation and development of BEC within our confined system, spanning a time range from 1 ns to 31 ns. At the outset, precisely at 1 ns, we witness the initial stages of BEC nucleation along the system's surface. This is characterised by the appearance of localised regions of condensed particles. These initial condensates are distributed randomly across the surface, reflecting the stochastic nature of the nucleation process in quantum systems. As we progress to 3 ns, a fascinating transformation occurs. The initially scattered condensates begin to self-organise, forming pairs that arrange themselves along a symmetric ring-shaped structure towards the center of the confined domain. This intriguing behaviour highlights the quantum nature of BEC and the subtle interplay of forces governing its dynamics. One noteworthy aspect of this simulation is the rate of change observed in these condensates and the damping effect experienced by the trapped BEC. This damping, which occurs over a time step of 2 ns, is indicative of the complex interactions within the system and the dissipation of energy over time. By 9 ns, the condensates have taken on a distinct ring-shaped pattern. This ring expands linearly with time, showcasing the dynamic nature of BEC growth within our confined system. This section provides a comprehensive examination of the evolution of BEC within a confined two-dimensional space. Through the numerical solution of the GP equation and the visualisation in Figure 3, we gain insights into the intricate processes of BEC nucleation, organisation, and expansion over time. This analysis contributes to a deeper understanding of quantum phenomena within confined systems and their implications for various scientific and technological applications.

IV. DISCUSSION

The control and manipulation of nonlinear dynamics within a system featuring finite-range interactions, especially within a confined space, are known to be challenging. The molecular dynamics within such systems often exhibit unpredictability, making it a complex task to understand and manage. However, the findings of our study shed light on a promising avenue for defining and manipulating nonlinear dynamics within confined domains. This novel finding is highly promising for applications spanning from the realm of nanofluidics to the far reaches of astrophysics. The ability to comprehend and control these dynamics within the constraints of a multi-dimensional Hilbert space represents a significant advancement in our understanding of natural phenomena. One notable aspect of our study is the concept of restructuring classical systems to yield desired observables by exerting control over their dynamics at the quantum level. This concept has the potential to serve as a powerful tool for scientists and researchers seeking to achieve specific outcomes in their experiments or simulations. The significance of the asymptotic function showcased in Figure 1.1 lies in its representation of the same soliton function as depicted in Figure 1.2, albeit with potential in the inverse Fourier-transformed space. This insight provides a valuable perspective on the interplay between different mathematical representations and their impact on the behaviour of soliton functions. A crucial observation is the disappearance of solitons at the 83 ns mark, which can be justified by mathematical limits. As the function approaches infinity, it does so with the condition $|\psi|^2$ approaching zero, resulting in the solitons vanishing from the system. The evolution of soliton waves, as evident in Figure 2, offers further insights. These soliton waves transition from their initial compact and pronounced forms to shallower and wider configurations as their dynamics evolve over time. This transformation is attributed to the influence of phase gradients on soliton wave dynamics. While some studies have demonstrated the generation of solitons by optically imprinting phase steps on condensate wave functions [38], this phenomenon is not replicated in the condensates observed in Figure 3. Consequently, the growth dynamics of these condensates were analysed, providing a time-dependent perspective on their development. By analysing and establishing specific operating conditions for these observed dynamics, we open up exciting possibilities, particularly in the realm of quantum computing. Our results may pave the way for quantum computers to efficiently solve nonlinear differential equations with unprecedented precision. Such advancements hold the potential to revolutionise fields reliant on complex mathematical modelling and simulations, ushering in a new era of computational capabilities. In conclusion, our study contributes to a deeper understanding of nonlinear dynamics within (nano)confined fluids and the potential for control and manipulation of the fluid dynamics at the quantum level.

AUTHORS' CONTRIBUTION

VJ performed the numerical computation. SC and SG collaborated on the concept and validated the work. SG developed the concept and supervised the research and secured funding for this research.

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