# Quantum Control of Nonlinear Dynamics in Confined Systems

Vinitha Johny[†] and Siddharth Ghosh[†‡*]
[†]*International Centre for Nanodevices, InCeNSE-TBI,*
*Indian Institute of Science Campus, Bangalore 560 012, Karnataka, India.*
[‡]*Department of Applied Mathematics and Theoretical Physics,*
*University of Cambridge, Cambridge CB3 0WA, UK.*

Investigating the intricacies of confined nonlinear dynamics presents formidable challenges, primarily due to the unpredictable behaviour of molecular constituents. This study introduces a promising avenue for comprehending and harnessing nonlinear dynamics within constrained domains, with broad applications spanning fields like nanofluidics and astrophysics. Quantum-level control emerges as a powerful tool, enabling the manipulation of classical systems to achieve specific outcomes, including quantum control of fluidic behaviour at the nanoscale for application in actuation in nanofluidics. Of particular significance is the observation of an asymptotic function that describes soliton behaviour within a transformed mathematical framework, shedding light on the practical implications of abstract representations. Solitons, known to vanish mathematically, exhibit intriguing transformations over time, influenced by phase gradients. Soliton formations, tracked from 1 ns to 83 ns, reveal dynamic transformations, evolving from their initial state with intriguing variations in amplitude and phase angle. These solitons, under the influence of subtle phase gradients, transition towards states characterised by reduced amplitude and expanded spatial extent. The ability to exercise quantum control over nanoscale fluidic behaviours beckons novel applications, notably in nanofluidic actuation. These findings hold the potential to revolutionise the efficiency of quantum computing in addressing nonlinear differential equations, offering new opportunities for precision-driven progress across scientific disciplines.

## I. INTRODUCTION

The intricate realm of nonlinear dynamics within confined systems has captivated us across disciplines, offering an intriguing intersection of physics, mathematics, and real-world applications [1–5]. In such systems, the behaviour of molecules and waves defies conventional expectations, often revealing hidden complexities that challenge our understanding [6, 7]. Within these confined domains, solitons, those elusive solitary waves characterised by their ability to maintain their shape as they propagate, have emerged as prominent actors [8–14]. Their extraordinary stability and widespread occurrence, from the control of mode-locked lasers to unravelling the secrets of fluid dynamics, have rendered them a captivating subject of investigation. This exploration offers a profound glimpse into a realm with far-reaching implications, spanning the controlled manipulation of fluidic behaviour at the nanoscale to resolving several research questions where confined systems underlie the very fabric of the cosmos. As we embark on this exploration of confined quantum domains, we find inspiration not only from the rich history of nonlinear physics but also from Chandrasekhar's groundbreaking work on the mathematical modelling of astrophysical singularities [15]. The complexities and paradoxes that nonlinear dynamics often present are reminiscent of the challenges Chandrasekhar grappled with [16], offering us a unique opportunity to bridge seemingly disparate disciplines.

The solution of nonlinear partial differential (NLPD) equations within confined spaces, particularly in the context of fluid dynamics, poses substantial computational challenges. These equations often feature hyperbolic and exponential terms that introduce complexities, potentially leading to inaccuracies and a weakening of the nonlinear component. Consequently, it becomes imperative to seek solution forms that offer enhanced stability, mitigating dispersion effects within the system [17]. Although various methodologies have been employed to tackle NLPD equations [18–20], there remains a notable gap in understanding the dependence of solutions on classical parameters, and visualisation of these solutions is frequently overlooked. One prominent domain where such nonlinear dynamics arise is the study of confined quantum systems, exemplified by trapped Bose-Einstein condensates. These systems, described using the Gross-Pitaevskii (GP) equation, exhibit many-body interactions in nanometric spaces. However, achieving such conditions often necessitates external boundary conditions, such as intense magnetic fields or extremely low temperatures [21]. The presence of multiple species with anisotropic properties in these short-range interactive spaces further amplifies the complexity of their dynamics.

The GP equation, which can be viewed as a specialised form of the nonlinear Schrödinger equation, accommodates an additional nonlinearity term to account for interspecies interactions:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + gN|\psi|^2\right) \quad (1)$$

Here, $V_{\text{ext}}$ represents the potential of the confined system, and $g = 4\pi\hbar^2(a/m)$, where $a$ is the scattering length. Additionally, we incorporate an extra term related to spins

* Email of correspondence: sg915@cam.ac.uk

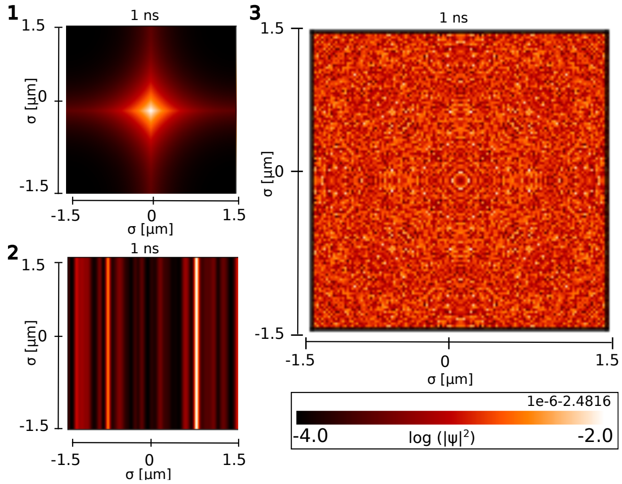


FIG. 1. (1) Evolution of the asymptotic curve in the numerical analysis of the 2D anisotropic Gross-Pitaevskii (GP) equation, independent of temperature, within a square domain of 3 μm in length. (2) Formation dynamics of observed solitons in the numerical analysis of the 2D anisotropic GP equation, accounting for temperature dependency, within a square domain of 3 μm in length. (3) Dynamics of observed condensates in the numerical analysis of the 2D anisotropic GP equation, considering temperature as a function of critical density, within a square domain of 3 μm in length.

to address the behaviour of vortices and predict the system's nonequilibrium dynamics more precisely and accurately [22, 23]. Notably, in systems featuring oscillator potentials, exact solutions for the Gross-Pitaevskii equation remain elusive [24]. Instead, solutions dynamically evolve with changes in time-dependent parameters. In this paper, we embark on a comprehensive exploration of these challenges and opportunities, presenting three distinct solutions to the GP equation within the confines of our research domain.

## II. THEORY

The exact solution to a Nonlinear Partial Differential Equation (NLPDE) is often constrained by specific operating conditions, which are intricately tied to the theoretical models governing the system and dependent on certain parameters [25]. Attempting to encapsulate the entire dynamics of such systems under these constraints is inherently complex. Dissipative solitons, on the other hand, emerge as a viable solution within nonlinear systems, owing to the interplay between dissipative and dispersive coefficients. These solitons exhibit remarkable stability and find utility in diverse applications, such as mode-locked lasers [26]. It is crucial to note that alterations in the aforementioned variables significantly influence the overall dynamics of a nonlinear system. In this discussion, we present a selection of exact solutions for nonlinear systems, drawing comparisons with Kudryashov's method for solving higher-order differential equations [27]. The influence of dispersive effects on the medium has been a subject of extensive research over the years. Lan et al. [28] have delved into variational formalism for dispersive equations within nonlinear media, revealing solitonic behaviour. Additionally, Karpman et al. [29], have examined the stability of soliton waves using the Lyapunov approach, a methodology that we also verify in our research, as illustrated in Figure 1.1. In the context of classical physics, solving equations of motion provides a comprehensive understanding of a system's dynamic variables at any given time, enabling the depiction of its complete behaviour. However, in the context of quantum mechanics, the equation of motion transforms into the variation of expectation values over time for the state vector within the abstract Hilbert space. We introduce a linear operator, represented by the unitary operator $\hat{U}$, specifically tailored to this quantum system. This operator accommodates the system's anisotropy along the x and y axes, while disregarding couplings along the z-axis. The wave function can be expressed as follows:

$$\psi(t) = \hat{U}(t, t_0)\psi(t_0) \tag{2}$$

Here, $\hat{U}$ is unitary ($U^\dagger U = 1$, $U^\dagger = U^{-1}$), and it corresponds to the system. Ensuring the normalization condition of $\psi$ ($\langle\psi(t)|\psi(t)\rangle = 1$) at the instants t and $t_0$ respectively, we have:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{3}$$

$$\langle\psi(t)| = \langle\psi(t_0)|U^\dagger(t, t_0) \tag{4}$$

We then define a Hermitian operator $\hat{H}$ corresponding to the unitary operator $\hat{U}$ as:

$$\hat{U} = \exp(i\alpha\hat{H}) \tag{5}$$

Here, $\alpha$ represents the parameter of change throughout the simulation. In a prior study, the annihilation operator influenced the flow-switching mechanism of nanofluidic pores, as eigenstates may not be analytical signals within the interaction space [30]. To investigate the time evolution of such a system, we assume that the state vector $\psi_H$ remains time-independent, while the spin operator $\hat{S}_H$ becomes time-dependent. Within Heisenberg's picture of the state vector, we have:

$$|\psi_H(t)\rangle = \hat{U}^{-1}(t, t_0)|\psi(t)\rangle \tag{6}$$

$$\hat{S}_H(t) = \hat{U}^{-1}(t, t_0)\hat{S}\hat{U}(t, t_0) \tag{7}$$

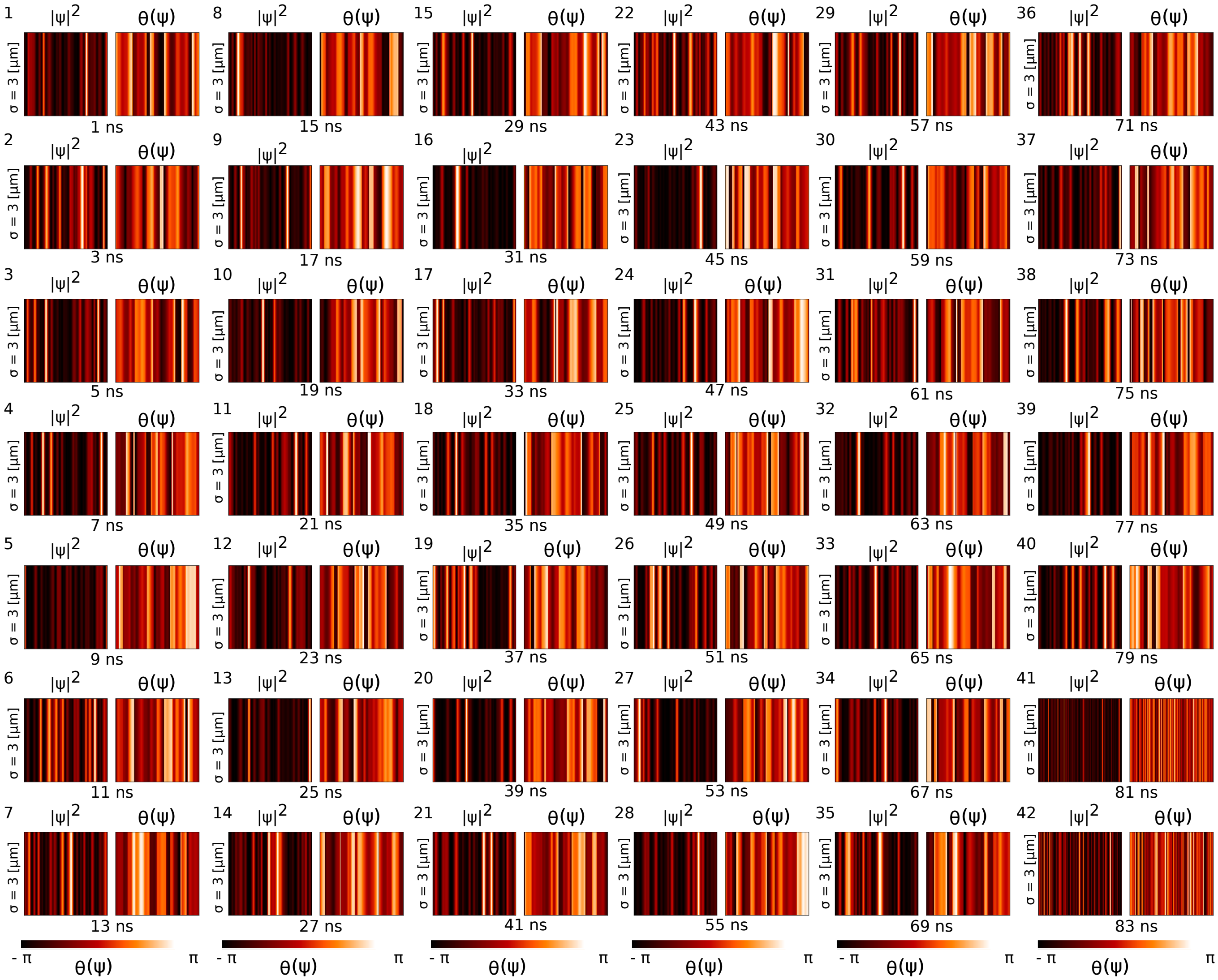


FIG. 2. Time evolution of soliton formation in a 2D anisotropic GP Equation (Hamiltonian in Equation 12) from 1 ns to 83 ns. Subfigures 1 to 42: Show the simultaneous plotting of the soliton's wave function amplitude ($|\psi|^2$) and phase angle ($\theta(\psi)$) with a 2 ns time step.

Now, assuming coherent states and a time-dependent spin operator governed by the dynamic variable $\theta$, Equation 5 becomes:

$$U^\dagger(\theta) S_i U(\theta) = \exp[-i, S_{ix}\theta] \tag{8}$$

$$U^\dagger(\theta) S_i U(\theta) = S_{ix} \tag{9}$$

$$U^\dagger(\theta) S_{iy} U(\theta) = S_i \cos\theta - S_{iz}\sin\theta \tag{10}$$

$$U^\dagger(\theta) S_{iz} U(\theta) = S_{iy}\sin\theta + S_{iz}\cos\theta \tag{11}$$

Given that we do not consider z-axis coupling interactions, the differential transformation along the x-axis is zero. The transformation along the y-axis can be expressed as: $\left.\frac{dS_{iy}}{d\theta}\right|_{\theta=0} = -S_{iz}$. Subsequently, the Hamiltonian of a system incorporating the spin operator is given by:

$$H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + \frac{i}{2}\left(\frac{P}{1+\frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_i}{d\theta}\right]\psi \tag{12}$$

$$\frac{dS_i}{d\theta} = U^\dagger(\theta) S_{ix} U(\theta) + U^\dagger(\theta) S_{iy} U(\theta) + U^\dagger(\theta) S_{iz} U(\theta) \tag{13}$$

As we do not consider z-axis coupling interactions, the above expression simplifies to $\frac{dS_i}{d\theta} = \frac{d}{d\theta}(S_i \cos\theta)$ Thus, Equation 12 becomes:

$$H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + \frac{i}{2}\left(\frac{P}{1+\frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_i \cos\theta}{d\theta}\right]\psi \tag{14}$$

## III. DYNAMICS IN A CONFINED DOMAIN

The time evolution of nonlinear dynamical systems often yields soliton solutions through numerical analysis. Some systems also exhibit vortices, spiral solutions, and asymptotic solutions. Figure 1.1 illustrates the temperature-independent solution of nonlinear dynamics within a confined space. In this simulation, we solve the Gross-Pitaevskii (GP) equation while considering the XY model within the Heisenberg picture. The solution assumes an asymptotic form at 1 ns, which eventually converges to a singular point by the end of 4 ns. Figure 1.2 presents the soliton solution within the same mathematical framework, albeit with potential expressed in inverse Fourier transform space. Figure 1.3 showcases the solution for the same Hamiltonian as before, but with additional parameters and operating conditions, including temperature and density of states, which are functionally dependent on geometry. This includes considering the relativistic wavelength and critical temperature as functions of the density of states.

### A. NLGP Solution to Solitons in Confined 2D Space

The stochastic nonlinear Schrödinger equation [31], delineates both the deterministic and stochastic components of the Hamiltonian. This Hamiltonian introduces quantum fluctuations (noise) in dispersive nonlinear systems, leading to the formation of quantum solitons. In such systems, nonlinearity effectively balances dispersion terms, resulting in the formation of nondispersive soliton waves [32]. We observe both bright and dark quantum solitons as solutions to the GP equation within the XY model. These observations occur over a confined square domain with a length of 3 µm. A bright soliton corresponds to a peak in the amplitude of the wavefunction, while a dark soliton exhibits a decrease in amplitude. The results are presented in Figure 2, covering a total period of 83 ns with a time step size of 2 ns.

In this simulation, the effective interaction between particles in the short-range interaction regime at the lowest energy range is denoted as $U = 4 \times \hbar^2 g/m$, where $U(\boldsymbol{r} - \boldsymbol{r_0})$ involves the position vectors $\boldsymbol{r}$ and $\boldsymbol{r_0}$ of two interacting particles. Additionally, the constants $m$ (representing the unitary mass of the particle), $P$ (saturation pumping strength of the particles) = 2, and system loss $q = 0.3$ are considered. The density is expressed as a function of the effective interaction between particles in a two-dimensional space:

$$n_0 = \rho \times \left(\frac{P}{q-1}\right) \tag{15}$$

Here, $\rho$ denotes the particle density, and $n_c$ is the critical density of the system. For a constant two-dimensional potential in Fourier-transformed space, the initial boundary conditions are $\psi = 1$, $\psi(t) = 1$, where $\psi(t)$ represents the wave function at time $t$. Several parameters vary in the simulation. The critical temperature ($T_c$) of the system, also a function of density, is given by: $T_c = 3.3\hbar^2\rho^{\frac{2}{3}}/mk_B$ The relativistic wavelength $\lambda$ is defined as: $\lambda = \frac{\hbar}{\sqrt{(4\pi m k_B T_c)}}$ The relativistic energy $E_n$ is expressed as: $E_n = 1.23/\lambda$ The chemical potential $Q$ is defined by: $Q = \frac{n_0 E_n - T k_B}{n_0}$

In Figure 2, we show a visual representation of the outcomes of our simulations, shedding light on the behaviour of bright and dark quantum solitons in a two-dimensional space within the system under investigation. These solitons, being prominent features of nonlinear systems, play a crucial role in understanding the system's dynamics. Bright solitons, characterised by peaks in the amplitude of the wavefunction, are observed in the figures as regions where the wavefunction amplitude remains relatively constant. Conversely, dark solitons manifest as areas where the amplitude of the wavefunction drops significantly. The presence and evolution of these solitons provide valuable insights into how the system responds to the interplay between nonlinearity and dispersion. As we progress through the figures, which span different time intervals, we witness a dynamic transformation in the solitons' patterns and interactions. At earlier time points, we observe isolated bright solitons with well-defined positions and amplitudes. As time advances, these solitons can merge, split, or even disappear, reflecting the intricate interplay between various factors influencing their behaviour. Additionally, we analyse the phase angles associated with these solitons. The phase angle of a soliton essentially describes the relative phase of its wavefunction. In the context of the figures, it tells us how the solitons' phases change spatially. The phase angle provides critical information about the solitons' stability and their interactions with other solitons or perturbations in the system.

### B. Evolution of NLGP Solution to Condensates in Confined 2D Space

We explore the dynamic growth of Bose-Einstein condensates (BEC), a unique quantum phenomenon, within a confined two-dimensional space. The mathematical framework for this exploration involves solving the Gross-Pitaevskii (GP) equation using the XY-model in the

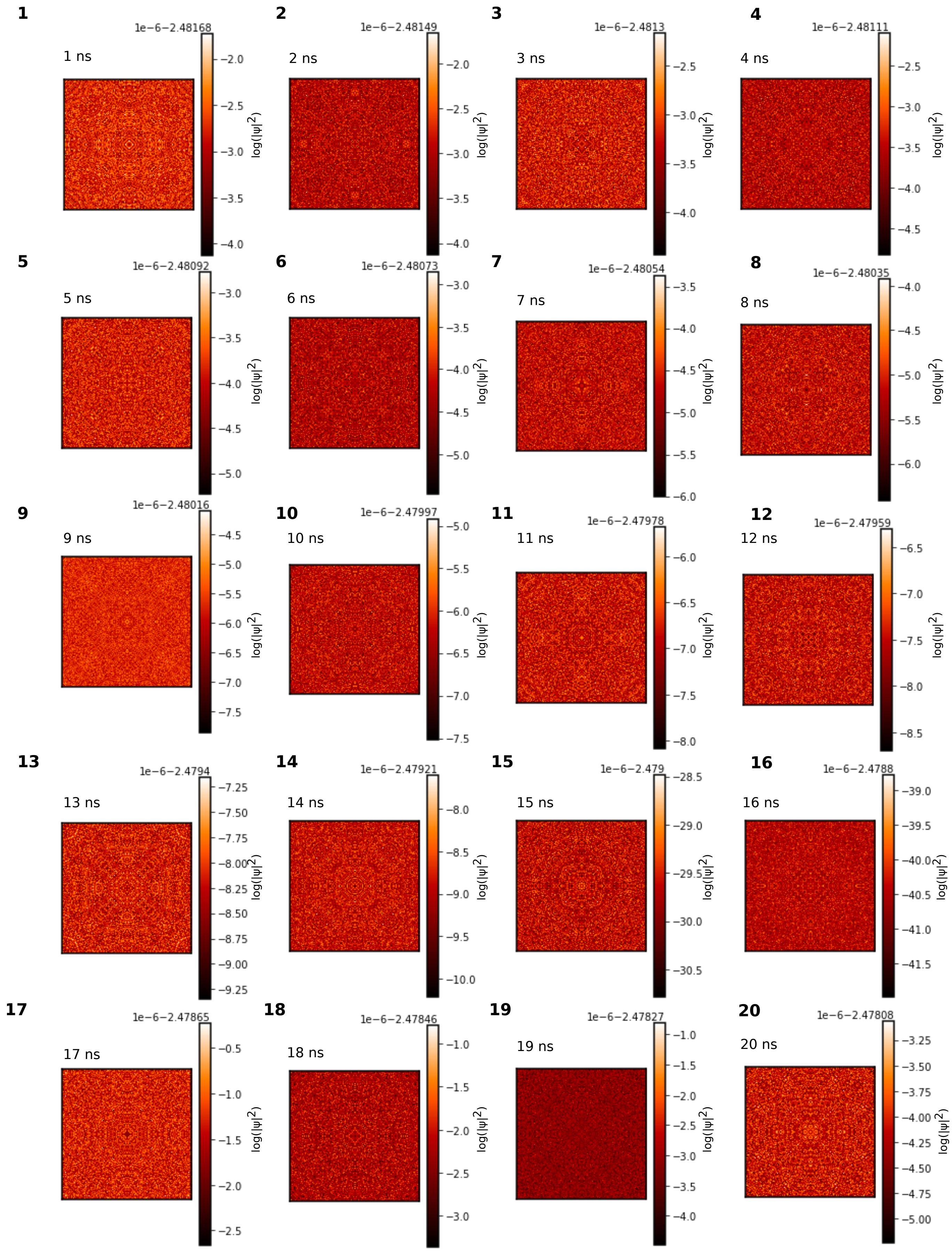


FIG. 3. 1 - 20: Growth dynamics and nucleation of BEC with time according to equation 14 – initial time frame = 1 ns with time steps of 1 ns and final time frame = 20 ns with time step = 1 ns.

Heisenberg picture, employing the numerical integration technique known as the Runge-Kutta method. The visual representation of these dynamics is depicted in Figure 3. Figure 3 provides a chronological overview of the nucleation and development of BEC within our confined system, spanning a time range from 1 ns to 31 ns. At the outset, precisely at 1 ns, we witness the initial stages of BEC nucleation along the system's surface. This is characterised by the appearance of localised regions of condensed particles. These initial condensates are distributed randomly across the surface, reflecting the stochastic nature of the nucleation process in quantum systems. As we progress to 3 ns, a fascinating transformation occurs. The initially scattered condensates begin to self-organise, forming pairs that arrange themselves along a symmetric ring-shaped structure towards the center of the confined domain. This intriguing behaviour highlights the quantum nature of BEC and the subtle interplay of forces governing its dynamics. One noteworthy aspect of this simulation is the rate of change observed in these condensates and the damping effect experienced by the trapped BEC. This damping, which occurs over a time step of 2 ns, is indicative of the complex interactions within the system and the dissipation of energy over time. By 9 ns, the condensates have taken on a distinct ring-shaped pattern. This ring expands linearly with time, showcasing the dynamic nature of BEC growth within our confined system. This section provides a comprehensive examination of the evolution of BEC within a confined two-dimensional space. Through the numerical solution of the GP equation and the visualisation in Figure 3, we gain insights into the intricate processes of BEC nucleation, organisation, and expansion over time. This analysis contributes to a deeper understanding of quantum phenomena within confined systems and their implications for various scientific and technological applications.

## IV. DISCUSSION

The control and manipulation of nonlinear dynamics within a system featuring finite-range interactions, especially within the confines of a confined space, are known to be challenging. The molecular dynamics within such systems often exhibit unpredictability, making it a complex task to understand and manage. However, the findings of our study shed light on a promising avenue for defining and manipulating nonlinear dynamics within confined domains. This revelation holds immense potential with applications spanning from the realm of nanofluidics to the far reaches of astrophysics. The ability to comprehend and control these dynamics within the constraints of a multi-dimensional Hilbert space represents a significant advancement in our understanding of natural phenomena. One notable aspect of our study is the concept of restructuring classical systems to yield desired observable by exerting control over their dynamics at the quantum level. This concept has the potential to serve as a powerful tool for scientists and researchers seeking to achieve specific outcomes in their experiments or simulations. The significance of the asymptotic function showcased in Figure 1.1 lies in its representation of the same soliton function as depicted in Figure 1.2, albeit with potential in the inverse Fourier-transformed space. This insight provides a valuable perspective on the interplay between different mathematical representations and their impact on the behaviour of soliton functions. A crucial observation is the disappearance of solitons at the 83 ns mark, which can be justified by mathematical limits. As the function approaches infinity, it does so with the condition $|\psi|^2$ approaching zero, resulting in the solitons vanishing from the system. The evolution of soliton waves, as evident in Figure 2, offers further insights. These soliton waves transition from their initial compact and pronounced forms to shallower and wider configurations as their dynamics evolve over time. This transformation is attributed to the influence of phase gradients on soliton wave dynamics. While some studies have demonstrated the generation of solitons by optically imprinting phase steps on condensate wave functions, this phenomenon is not replicated in the condensates observed in Figure 3. Consequently, the growth dynamics of these condensates were analysed, providing a time-dependent perspective on their development. By analysing and establishing specific operating conditions for these observed dynamics, we open up exciting possibilities, particularly in the realm of quantum computing. These insights may pave the way for quantum computers to efficiently solve nonlinear differential equations with unprecedented precision. Such advancements hold the potential to revolutionise fields reliant on complex mathematical modelling and simulations, ushering in a new era of computational capabilities. In conclusion, our study contributes to a deeper understanding of nonlinear dynamics within confined systems and the potential for control and manipulation at the quantum level. These findings not only enrich our scientific knowledge but also offer practical applications that can have a profound impact on various scientific disciplines.

## ACKNOWLEDGMENTS

The VJ and SG thank the Honeywell CSR fund and K-Tech funding from the Government of Karnataka awarded to the International Center for Nano Devices. SG thanks the German Research Foundation and Isaac Newton Trust for funding his research. The authors thank Professor Natalia Berloff for many productive discussions.

[1] S. Savasta and R. Girlanda, Quantum optical effects and nonlinear dynamics in interacting electron systems, Physical review letters **77**, 4736 (1996).

[2] L. H. Olesen, M. Z. Bazant, and H. Bruus, Strongly nonlinear dynamics of electrolytes in large ac voltages, Physical Review E **82**, 011501 (2010).

[3] D. B. Brückner, A. Fink, C. Schreiber, P. J. Röttgermann, J. O. Rädler, and C. P. Broedersz, Stochastic nonlinear dynamics of confined cell migration in two-state systems, Nature Physics **15**, 595 (2019).

[4] A. Engel, G. Smith, and S. E. Parker, Linear embedding of nonlinear dynamical systems and prospects for efficient quantum algorithms, Physics of Plasmas **28** (2021).

[5] T. Brandstätter, D. B. Brückner, Y. L. Han, R. Alert, M. Guo, and C. P. Broedersz, Curvature induces active velocity waves in rotating spherical tissues, Nature Communications **14**, 1643 (2023).

[6] V. Johny and S. Ghosh, Self-driven flow and chaos at liquid-gas nanofluidic interface, arXiv preprint arXiv:2112.15220 (2021).

[7] V. Johny, S. A. Conter, and S. Ghosh, Superfluidic nature of self-driven nanofluidics at liquid-gas interfaces, arXiv preprint arXiv:2208.13759 (2022).

[8] L. Salasnich, A. Parola, and L. Reatto, Modulational instability and complex dynamics of confined matter-wave solitons, Physical review letters **91**, 080405 (2003).

[9] J. H. Nguyen, P. Dyke, D. Luo, B. A. Malomed, and R. G. Hulet, Collisions of matter-wave solitons, Nature Physics **10**, 918 (2014).

[10] S. Das, S. Roh, N. Atzin, A. Mozaffari, X. Tang, J. J. de Pablo, and N. L. Abbott, Programming solitons in liquid crystals using surface chemistry, Langmuir **38**, 3575 (2022).

[11] G. W. Henderson, G. R. Robb, G.-L. Oppo, and A. M. Yao, Control of light-atom solitons and atomic transport by optical vortex beams propagating through a bose-einstein condensate, Physical Review Letters **129**, 073902 (2022).

[12] G. Wang, K. Hou, Y. Liu, H. Bi, W. Li, and Y. Xue, Controllable bistability and squeezing of confined polariton dark solitons, Optics Express **31**, 22722 (2023).

[13] J. Mäkinen, K. Zhang, and V. Eltsov, Vortex-bound solitons in topological superfluid 3he, Journal of Physics: Condensed Matter **35**, 214001 (2023).

[14] A. Roy and S. Lukyanov, Soliton confinement in a quantum circuit, arXiv preprint arXiv:2302.06289 (2023).

[15] S. Chandrasekhar and B. C. Xanthopoulos, On colliding waves that develop time–like singularities: a new class of solutions of the einstein–maxwell equations, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **410**, 311 (1987).

[16] S. Chandrasekhar, On stars, their evolution and their stability, Science **226**, 497 (1984).

[17] A. Ripai, T. E. Sutantyo, Z. Abdullah, M. Syafwan, and W. Hidayat, Effect of ansatz on soliton propagation pattern in photorefractive crystals, in *Journal of Physics: Conference Series*, Vol. 1876 (IOP Publishing, 2021) p. 012009.

[18] V. N. Serkin and A. Hasegawa, Novel soliton solutions of the nonlinear schrödinger equation model, Physical Review Letters **85**, 4502 (2000).

[19] S. Chávez Cerda, S. B. Cavalcanti, and J. Hickmann, A variational approach of nonlinear dissipative pulse propagation, The European Physical Journal D-Atomic, Molecular, Optical and Plasma Physics **1**, 313 (1998).

[20] S. Roy and S. K. Bhadra, Solving soliton perturbation problems by introducing rayleigh's dissipation function, Journal of lightwave technology **26**, 2301 (2008).

[21] L. Erdős, B. Schlein, and H.-T. Yau, Rigorous derivation of the gross-pitaevskii equation, Physical review letters **98**, 040404 (2007).

[22] R. Sensarma, M. Randeria, and T.-L. Ho, Vortices in superfluid fermi gases through the bec to bcs crossover, Physical review letters **96**, 090403 (2006).

[23] M. J. Ku, W. Ji, B. Mukherjee, E. Guardado-Sanchez, L. W. Cheuk, T. Yefsah, and M. W. Zwierlein, Motion of a solitonic vortex in the bec-bcs crossover, Physical review letters **113**, 065301 (2014).

[24] R. Atre, P. K. Panigrahi, and G. S. Agarwal, Class of solitary wave solutions of the one-dimensional gross-pitaevskii equation, Physical Review E **73**, 056611 (2006).

[25] N. A. Kudryashov, Method for finding highly dispersive optical solitons of nonlinear differential equations, Optik **206**, 163550 (2020).

[26] K. Krupa, K. Nithyanandan, U. Andral, P. Tchofo-Dinda, and P. Grelu, Real-time observation of internal motion within ultrafast dissipative optical soliton molecules, Physical review letters **118**, 243901 (2017).

[27] M. Mirzazadeh, M. Eslami, and A. Biswas, Dispersive optical solitons by kudryashov's method, Optik **125**, 6874 (2014).

[28] Z.-Z. Lan, Multi-soliton solutions for a (2+ 1)-dimensional variable-coefficient nonlinear schrödinger equation, Applied Mathematics Letters **86**, 243 (2018).

[29] V. Karpman, Lyapunov approach to the soliton stability in highly dispersive systems. ii. kdv-type equations, Physics Letters A **215**, 257 (1996).

[30] C. Mehta and E. Sudarshan, Time evolution of coherent states, Physics Letters **22**, 574 (1966).

[31] S. Carter, P. Drummond, M. Reid, and R. M. Shelby, Squeezing of quantum solitons, Physical review letters **58**, 1841 (1987).

[32] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, *et al.*, Generating solitons by phase engineering of a bose-einstein condensate, Science **287**, 97 (2000).

[33] L. Erdos, B. Schlein, and H.-T. Yau, Derivation of the gross-pitaevskii equation for the dynamics of bose-einstein condensate, arXiv preprint math-ph/0606017 (2006).

[34] J. Rogel-Salazar, The gross–pitaevskii equation and bose–einstein condensates, European Journal of Physics **34**, 247 (2013).

[35] M. Wadati, T. Iizuka, and M. Hisakado, A coupled nonlinear schrödinger equation and optical solitons, Journal of the Physical Society of Japan **61**, 2241 (1992).

[36] Q. Chang, E. Jia, and W. Sun, Difference schemes for solving the generalized nonlinear schrödinger equation, Journal of Computational Physics **148**, 397 (1999).

[37] J. Sierra, A. Kasimov, P. Markowich, and R.-M. Weishäupl, On the gross–pitaevskii equation with pumping and decay: stationary states and their stability, Journal of Nonlinear Science **25**, 709 (2015).

[38] E. J. Madarassy and V. T. Toth, Numerical simulation code for self-gravitating bose–einstein condensates, Computer Physics Communications **184**, 1339 (2013).

[39] J. Chwedeńczuk, P. Ziń, K. Rza̧żewski, and M. Trippenbach, Simulation of a single collision of two bose-einstein condensates, Physical review letters **97**, 170404 (2006).

[40] S. Eckel, A. Kumar, T. Jacobson, I. B. Spielman, and G. K. Campbell, A rapidly expanding bose-einstein condensate: an expanding universe in the lab, Physical Review X **8**, 021021 (2018).

[41] S. Choi, S. Morgan, and K. Burnett, Phenomenological damping in trapped atomic bose-einstein condensates, Physical Review A **57**, 4057 (1998).

[42] H. Wang, Q. Zhou, A. Biswas, and W. Liu, Localized waves and mixed interaction solutions with dynamical analysis to the gross–pitaevskii equation in the bose–einstein condensate, Nonlinear Dynamics **106**, 841 (2021).

[43] L. Xu, Variational approach to solitons of nonlinear dispersive k (m, n) equations, Chaos, Solitons & Fractals **37**, 137 (2008).