Nanoscale anisotropic dynamical systems are tedious to solve in a classical domain without any approximation techniques. In such cases, the interactive space and force are significant in defining their motions. We solve and simulate the phase transition motion of water in the interface of 3 μm × 3 μm confined water domain. We demonstrate the presence of solitons in Bose-Einstein condensate at nanosecond time scale by solving a nonlinear partial differential equation. We find an exact solution of the 2D Gross-Pitaevskii equation with XY-model using the Heisenberg picture which creates solitons and condensate nucleation.

I. INTRODUCTION

The time evolution of nonlinear partial differential (NLPDE) equations is quite demanding to be solved for certain fluid dynamics contexts in a confined space. Often in some domains, the hyperbolic and exponential terms in the governing equations may cause inaccuracies and the nonlinear component becomes weak. Hence, it creates solution forms of higher stability — reduced resistance to dispersion in the system [1]. Although a wide range of approaches has been adapted to solve such NLPD [2–4], the dependencies of their solution on certain classical parameters are not explored. But often the visualisation of these solutions are missed out. The many-body interaction in confined nanometric space containing one or more species is a common phenomenon in trapped Bose-Einstein condensates and is described by advanced level of quantum statistics using the Gross-Pitaevskii (GP) equation. But the occurrence of this phenomenon usually demands external boundary conditions, such as strong magnetic fields or low temperatures [5]. The anisotropy of the species in a short-range interactive space makes such systems and their dynamics more complex. We can consider the GP equation as a special case of a nonlinear Schrödinger equation which incorporates an additional term of nonlinearity which takes into account the interaction between species in the system [6]. The Schrödinger equation is given by

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{ext} \right) \psi. \]

Considering the nonlinear term it becomes,

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{ext} + g|\psi|^2 \right) \]

where \( V_{ext} \) is the potential of confined system, \( g = 4\pi\hbar^2(a/m) \), where \( a \) is the scattering length. Now, we incorporate one more additional term related to spins for vortices to predict the nonequilibrium dynamics of the system more precisely and accurately [7, 8]. In a system with oscillator potential, there is no exact solution for the Gross-Pitaevskii equation [9], and it will vary with the time-dependent parameters of the system. In this paper, we present three different solutions to the GP equation in a confined domain.

II. THEORY

The exact solution to an NLPDE is generally constrained by certain operating conditions, bound to the theoretical models defining the system which is a function of certain parameters [10]. It is complex to fit in the whole dynamics of such systems under this circumstance. Dissipative solitons can be generated and persisted through energy exchange in nonlinear systems with respect to the dissipative and dispersive coefficients. These solitons have significant stability and can be useful in many applications such as in mode-locked lasers [11]. Theoretically, the change in the above variables is crucial in determining the whole dynamics of a nonlinear system. Let us discuss a few exact solutions of nonlinear systems in contrast to Kudryashov’s method to solve higher order differential equations [12]. The existence of dispersive

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effects on the medium can bring certain predictable nonlinearity in the system which has been widely studied throughout the years. Lan et al. have solved variational formalism for dispersive equations of nonlinear medium and showed solitons [13]. Karpman et al. studied the stability of soliton waves using the Lyapunov approach [14] which is also verified in our case in Figure 1.1.

It is well known that solving the equations of motion in a classical context gives us the dynamical variable at any instant of time which can draw the complete dynamical behaviour of the system. Whereas in quantum mechanics, the equation of motion becomes the variation of expectation values with time for the state vector in the abstract Hilbert space. Let us define a linear operator corresponding to this system in terms of a unitary operator given by $\hat{U}$. The anisotropy of the system is considered only on the $x$ and $y$ axis and all the couplings on the $z$-axis are not considered. The wave function can be given as

$$\psi(t) = \hat{U}(t, t_0)\psi(t_0)$$

(1)

where $\hat{U}$ is unitary ($U^\dagger U = 1, U^\dagger = U^{-1}$) which corresponds to the system. By normalizing condition of $\psi$, $(\langle \psi(t) | \psi(t) \rangle = 1$ and, $(\langle \psi(t_0) | \psi(t_0) \rangle = 1$) at the instant $t$ and $t_0$ respectively.

$$|\psi(t)| = |\psi(t_0)|$$

(2)

Now, we define a hermitian operator $\hat{H}$ corresponding to the unitary operator $\hat{U}$ by

$$\hat{U} = \exp[i\alpha \hat{H}]$$

(4)
where $\alpha$ is the parameter of change throughout the simulation. In our previous study, the annihilation operator influenced the flow-switching mechanism of nanofluidic pores as eigenstates may not be analytical signals in the interaction space [15]. To study the time evolution of such a system, we choose the state vector $\psi_H$ to be time-independent and that of the spin operator $S_H$ to be time dependent. By Heisenberg’s picture of the state vector, 
\[
|\psi_H(t)\rangle = \hat{U}^{-1}(t,t_0)|\psi(t)\rangle \\
\hat{S}_H(t) = \hat{U}^{-1}(t,t_0)\hat{S}\hat{U}(t,t_0)
\]
Now, we consider the states to be coherent and the spin operator depends on the dynamic variable $\theta$. Then, equation 4 becomes,
\[
U^\dagger(\theta)S_1U(\theta) = \exp[-i, S_{1z}\theta] 
\]
\[
U^\dagger(\theta)S_1U(\theta) = S_{1x}, 
\]
\[
U^\dagger(\theta)S_1U(\theta) = S_1\cos \theta - S_{1z}\sin \theta 
\]
\[
U^\dagger(\theta)S_{1z}U(\theta) = S_{1y}\sin \theta + S_{1z}\cos \theta 
\]
Since we do not consider the z-axis coupling interactions, the differential transformation in the x-axis is 0. we consider the transformation in the y-axis, to be as $\frac{dS_{1y}}{d\theta}|_{\theta=0} = -S_{1z}$. Now, the hamiltonian of a system incorporating the spin operator is given by,
\[
H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + i\frac{1}{2} \left(\frac{P}{1 + \frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_1}{d\theta}\right] \psi 
\]
\[
\frac{dS_1}{d\theta} = U^\dagger(\theta)S_{1x}U(\theta) + U^\dagger(\theta)S_{1y}U(\theta) + U^\dagger(\theta)S_{1z}U(\theta) 
\]
Since the coupling interaction in the z-axis is not considered, the above expression reduces to, $\frac{dS_1}{d\theta} = \frac{dS_1}{d\theta}(S_1\cos \theta)$ then equation 11 becomes,
\[
H(t) = \left[\frac{-\hbar^2\nabla^2}{2m} + g|\psi|^2 + i\frac{1}{2} \left(\frac{P}{1 + \frac{|\psi|^2}{n_s}} - \gamma\right) + \frac{dS_1\cos \theta}{d\theta}\right] \psi 
\]

### III. DYNAMICS IN A CONFINED DOMAIN

The time evolution of nonlinear dynamical systems often has soliton solutions by its numerical analysis. Some systems also have vortices and spiral solutions as well as asymptotic solutions. Figure 1.1 shows the temperature-independent solution of the nonlinear dynamics of a confined space. Here, the simulation solves the GP equation in light of the XY model in the Heisenberg picture. The solution takes an asymptotic form at 1 ns which later converges to a singular point by the end of 4 ns. Now, Figure 1.2 shows the soliton solution within the same mathematical framework but, with potential in inverse Fourier transform space. Figure 1.3 shows the solution to the same Hamiltonian as earlier with some additional parameters and operating conditions such as temperature and density of states with a functional dependency on the geometry. Here, relativistic wavelength along with the critical temperature as a function of the density of states are taken into consideration.

#### A. NLGP solution to solitons in confined 2D space

The quantum theory is based on the stochastic nonlinear Schrödinger equation in [16] have depicted the deterministic part and stochastic part of the Hamiltonian which is a source of quantum fluctuations (noise) in a dispersive nonlinear system which leads to the formation of quantum solitons. The nonlinearity balances dispersion terms in such systems to form nondispersive soliton waves [17]. We observe bright and dark quantum solitons as the solution to the GP equation in the XY-classical model with time within a confined square domain of length 3 μm. A bright soliton represents a peak in the amplitude of the wavefunction whereas a dark soliton shows a plummet. The results observed are shown in Figure 2 for a total period of 83 ns with a step time size of 2 ns. Now, the effective interaction between the particles within the short-range interaction at the lowest energy range is taken to be $U = 4 \times \hbar^2 q/m$ with a finite short-range interaction of $U(r - r_0)$ where $r$ and $r_0$ are the position vectors of two interacting particles under consideration. The constants $m$ is a unitary mass of the particle. The saturation pumping strength of the particles [18] within the domain is considered to be $P = 2$, and the loss of the system is taken to be $q = 0.3$. The density is given as a function of effective interaction between the particles spanning along the two-dimensional space. Then,
\[
n_0 = \rho \times \left(\frac{P}{q - 1}\right) 
\]
where $\rho$ is the particle density and $n_c$ is the critical density of the system. For a constant two-dimensional potential in the Fourier transformed space, the initial boundary conditions are given by, $\psi = 1$, $\psi(t) = 1$ where $\psi(t)$ is the wave function after the time $t$.

The varying parameters in the simulation are, the critical temperature ($T_c$) of the system which is also a function of density and is given by,
\[
T_c = 3.3\hbar^2 \rho_{1/3} / mk_B
\]
FIG. 3. 1 - 20: Growth dynamics and nucleation of BEC with time in accordance to equation 13 – initial time frame =1 ns with time steps of 1 ns and final time frame= 20 ns with time step = 1 ns.
Then the relativistic energy \( E_n \) is given by,
\[
E_n = \frac{1.23}{\lambda}
\]
(17)

In such a system, the chemical potential is defined by,
\[
Q = \frac{n_0E_n - Tk_B}{n_0}
\]
(18)

In Figure 2, the observed result of bright and dark quantum solitons along with their phase angle in two-dimensional space in the system under consideration is shown. In Figure 2.1, the bright soliton (constant amplitude) has the most stable modulation along 2 \( \mu \)m and relatively lower stability in the same 1 \( \mu \)m. In the case of its phase angle, these two amplitudes are anti-symmetric that is \( \pi \) and \(-\pi\) respectively. At 3 ns in Figure 2.2, brighter solitons have appeared with prominent stability at 2 \( \mu \)m as in the previous case. At 5 ns in Figure 2.3, now the brightest soliton with at most stable amplitude has shifted to \( \sigma = 1 \mu \)m. At 7 ns in Figure 2.4, there are three soliton waves with similar amplitudes distributed randomly at \( \sigma = 0.7, 1, 2.9 \) respectively. At 9 ns in Figure 2.5, there are no bright solitons as in earlier cases. At 11 ns in Figure 2.6, the bright solitons are observed again and get prominent towards the centre region at 13 ns in Figure 2.7. At 15 ns in Figure 2.8 there is only a single prominent soliton wave at \( \sigma = 1 \mu \)m and direction of propagation is not clear. This single soliton wave gets split into 2 at 17 ns in Figure 2.9. They come further at proximity within 0.5 \( \mu \)m at 19 ns in Figure 2.10. Now the collision gets higher with time as there are no more stable solitons in the system at 21 ns in Figure 2.11. At 23 ns in Figure 2.12, wave are self strengthened to form 2 bright solitons at \( \sigma = 1 \mu \)m, 2 \( \mu \)m. At 25 ns in Figure 2.13, the soliton waves have moved apart to positions \( \sigma = 0.9 \mu \)m, 2.9 \( \mu \)m respectively. At 27 ns in Figure 2.14, the solitons are at proximity as in Figure 2.10. Here they are brighter, hence highly stable than earlier. At 29 ns in Figure 2.15, random distribution of solitons is observed with varying brightness. In Figure 2.16 - 2.22, the solitons do not follow a trend to consider their dynamics predictable. After a periodic interval, at 23 ns in Figure 2.23, only a single soliton wave with the highest stability is observed. In Figures 2.24 - 2.40, the positions of varying bright and dark solitons are random. In Figures 2.41 the collisions increase to a greater extent with dark soliton number exceeding the bright solitons. This trend increases further at 83 ns, where the solitons disappear and the amplitude of the wave function diminishes. After 83 ns, the amplitude of the wave functions goes to infinity and hence no solitons are observed after Figures 2.42. Now, the existence of these localized soliton waves alone does not guarantee the possibility of no other interaction in the system. Hence the coexistence of mixed interaction solutions along with the localized soliton waves [19] has to be explored further.

B. Evolution of NLGP solution to condensates in confined 2D space

The growth dynamics of Bose-Einstein condensates observed when solving the GP equation with XY-model in the Heisenberg picture by the Runge-Kutta method is given in Figure 3. The nucleation of the BEC at 512 spatial resolution in a confined two-dimensional domain for 1 ns to 31 ns is shown in Figure 3.1 - 3.16. At 1 ns, the initiation of nucleation of the condensates is observed along the surface. There are also randomly distributed condensates on the surface. At 3 ns, these nucleates arrange themselves in a completely different orientation in pairs along a symmetric ring-shaped structure to the centre. Even though the overall dynamics of these condensates are similar to that of previous research [20–22], the rate of change in these condensates and damping of trapped BEC [23] for a time step size of 2 ns is evident here. At 9 ns, the condensates are ring-shaped which is linearly expanded with an increase in time.

IV. DISCUSSION

The control mechanism of nonlinear dynamics within finite range interactions of a confined system is tedious as its molecular dynamics are often unpredictable. Our study reveals the possibility of defining and manipulating the nonlinear dynamics in a confined domain to a certain extent which can have wide applications from nanofluidics to astrophysics. Understanding and defining such dynamics in nature within the limits of \( n \)-dimensional Hilbert space is quite demanding in the present context. Hence, restructuring the classical system under consideration to give desired observable by controlling its dynamics in the quantum realm can be a useful tool. The significance of the asymptotic function in Figure 1.1 is that it represents the same soliton function as in Figure 1.2 with potential in inverse Fourier transformed space. The dynamics of soliton disappearing at 83 ns is justified for its mathematical limits – function approaches infinity as \( |\psi|^2 = 0 \). As observed in Figure 2, the soliton waves become shallow and wider as their dynamics change. This is due to the influence of phase gradient on the soliton wave dynamics. Denschlag et al. optically imprinted phase step on the condensate wave function which could generate a soliton [17]. However, this is not the case for condensates observed in Figure 3. Hence, the growth dynamics were analysed and obtained for a given time with a definite time step size. Analysing and setting specific operating conditions for these observed dynamics can give clear pathways for quantum computers to solve nonlinear differential equations in a highly efficient manner with
greater precision.

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