A Quantum Information Processing Machine for Computing by Observables

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Abstract (222 words)
A quantum machine that accepts an input and processes it in parallel is described. The logic variables of the machine are not wavefunctions (qubits) but observables (i.e., operators) and its operation is described in the Heisenberg picture. The active core is a solid-state assembly of small nanosized colloidal quantum dots (QDs) or dimers of dots. The size dispersion of the nanodots that causes fluctuations in their discrete electronic energies is a limiting factor. The input to the machine is provided by a train of very short in time laser pulses, at least four in number. The coherent bandwidth of each ultrashort pulse needs to span at least several and preferably all the single electron excited states of the dots. The spectrum of the QD assembly is measured as a function of the time delays between the input laser pulses. The dependence of the spectrum on the time delays is Fourier transformed to a frequency spectrum. This spectrum of a finite range in time is made up of discrete pixels. These are the visible, raw, basic logic variables. The spectrum is analyzed to determine a possibly smaller number of principal components. A Lie-algebraic point of view is used to explore the use of the machine to emulate the dynamics of other quantum systems. An explicit example demonstrates the considerable quantum advantage of our scheme.
**Introduction**

Ultrafast spectroscopy is a major direction to novel quantum technologies (1, 2). Quantum nonlinear spectroscopy (3) offers many ways to control quantum systems and to take advantage of the superposition that is inherent to quantum dynamics (4). An extensively used nonlinear technique is the two dimensional, 2D, spectroscopy that was first explored in the infrared and has since been usefully applied in the UV (5-7). It is an ultrafast technique because the light pulses used must be short enough that their coherent width in energy can cover the span of several quantum states. Thereby one can prepare a coherent superposition of states. Experimentally it is now recognized that even in the UV the coherence in 2D electronic spectroscopy can survive for more than a few periods of fast beating electronic coherences, with periods as short as a dozen of fs.(2) There is therefore a time window where the electronic degrees of freedom are not yet effectively coupled to the intrasystem vibrations and neither to the environment. Any loss of coherence is then primarily inhomogeneous in origin. 2D spectroscopy has done much towards the understanding of the dynamics of proteins and other larger molecules that are of interest in biology.(8, 9) Inorganic systems that have a higher density of electronic states in the visible and near UV are quantum dots (QD).(10-12) We specifically mean colloidal quantum dots that are molecular systems prepared by chemical means, systems whose mean size can be small, say a very few nanometers. The price for this small size is the dispersion in size which, when limited, is the main source of inhomogeneous dephasing of the coherence of an ensemble of dots.(13) The design that we discuss has an essential difference from the canonical formulations of quantum computing(14) and/or quantum machine learning(15-17). The familiar quantum gates operates on wave functions. The elementary unit is a qubit, a linear combination of two states. A Qudit is a coherent combination of several states.(18-20) Here we work with observables, typically Hermitian operators that are said to live in Liouville space(21) unlike wave functions whose habitat is the Hilbert space. Very early on Dirac established the equivalence of the two descriptions. We shall try to highlight the advantage of designing with observables. Of course, for such applications where a wave function description is currently very highly refined such as in quantum chemistry, computing an electronic wave function is a highly worth-while goal. This is so even when we recognize that a wave function contains much too much detail than needed for determining the usual observable properties. There is also the very practical aspect of the enormous resources needed to store a wave function for a not small many body system let alone a protein.
For people interested in dynamics and in a time dependent view of spectroscopy (1, 3, 22, 23) an approach based on the Heisenberg picture dynamics of observables can be of more direct interest. We here examine in detail the spectral maps that are generated by 2D electronic spectroscopy of small quantum dots and dimers of small dots. There is combined experimental and theoretical evidence that for small few nm dots connected by small sub nanometric linkers there is coherent coupling between the two dots in the dimer. (10-12, 24) We use a boxCARS geometry (25) to describe the experiment because it is simpler and sufficient for our purpose. There are clear advantages for using action detection such as fluorescence (26) or photocurrents (27-29) and for using a larger number of laser pulses (6, 8 or even more) as compared to the four pulses that are used in current 2D scheme. However, we mean to demonstrate that already the simpler approach offers what we ultimately seek namely to claim that that the hardware combined with a 2D spectroscopy can act as a quantum computer in the original sense of Feynman (30). We have two central technical aims. The first is to characterize the spectral maps produced by the quantum dot arrays assemblies that are used in the actual device. This is analogous to seeking to determining the process matrix in the sense of Yuen-Zhou et al. (1, 31) The fast dephasing of the excited states means that experimentally one is limited to having only the ground electronic state of the dots as an initial state. So we can determine a row of the process matrix. The 2D experimental results of our hardware are conveniently represented as spectral maps. A spectral map is the polarization of the density matrix at a series of time points. (32) We begin by first vectorizing the maps at a sequence of time points along the T2 axis, see Figure 1. Each map is thereby a column of the ‘overall’ matrix representing the entire output. We aim to achieve a major compaction of the experimental results by performing a singular value decomposition, SVD, on the overall matrix and retaining only the more major components.

Our second central aim is to show how the dynamics of the hardware enables us to compute or, strictly speaking, to emulate, the dynamics of a different, other, quantum system. This targeted system is the system of interest. There are, of course, limitations. First and foremost the (finite) size of the Hilbert space in which the system of interest is specified needs to be smaller or equal to the size, N, of the Hilbert space that is sufficient to describe the (short time) dynamics of the hardware. Of course, we need to specify the system of interest and for this purpose we will require the finite spectrum of its Hamiltonian or a set of operators (= the observables) that is closed under commutation with one another and with the Hamiltonian. See refs. (33, 34) for simple concrete
examples. In our Hilbert space a sufficient set of operators that meet both requirements is the set of Gelfand projectors, $E_{ij}$, that are represented as $N$ by $N$ matrices where all entries are zero except in the row $i$ and column $j$ where the value is unity. $N$ is the size of the relevant Hilbert space and any matrix in this Hilbert space can be written as a linear combination of these $N^2$ Gelfand matrices. For the theorist the maps produced by the 2D spectroscopy are $N$ by $N$ matrices with entries, the polarizations, see below, that are complex numbers. Similarly the outputs of the emulation of the system of interest are such matrices. To make these comments more concrete we begin by discussing the hardware and its spectroscopy as determined by the 2D experimental scheme. This will bring forward the point that for the experimentalists the maps produced by the 2D spectroscopy are typically larger than the $N$ by $N$ matrices of the theorist. The entries in the directly observed experimental maps are the intensities observed after a Fourier transforming the emitted signal as a function of time intervals, see figure 1 below, to frequency axes. The overall range in time of the intervals determines the width in frequency of the individual pixels while the spacing in time points determines the number of pixels along a frequency axis. Typically there are many more than $N^2$ pixels in any experimental 2D frequency map. For a theorist, a map of a physical system at a given time is an $N$ by $N$ matrix. As is discussed in detail below, the larger number of pixels that is addressed can be made into an advantage in terms of how many different initial states for the emulation can be prepared in just one experiment.

The Hardware
We consider an assembly of dimers of coherently coupled quantum dots. The number $N$ is the number of one photon excited states that are accessible from the ground state, GS. See (11, 13) for concrete examples and details. These one-photon excited states are sometimes called, by analogy, excitons. We here deal exclusively with monoexcitons but towards achieving higher information processing and storage capabilities it can be of interest to access also higher, (biexciton etc), excited states.
Figure 1: (a) The sequence of 3 pulses used in a BoxCARS measurement and the 3 time intervals. (b) The generation of the polarization cube, measured as a function of the time intervals (left). After Fourier transformation of the $T_1$ and the $T_3$ axis (right), adapted from ref. (12), the cube is converted, (c) to a series of $N_{T3}$ by $N_{T1}$ frequency maps at $N_{T2}$ successive values of $T_2$. (c) Scheme of the vectorization of the frequency maps. Each map is converted to a column vector and these vectors are arranged as $N_{T2}$ columns of a matrix of $N_{T3}$ by $N_{T1}$ rows.

The spectroscopy as discussed here is performed in a BoxCARS geometry using a sequence of three short fs laser pulses that are not overlapping in time, figure 1. One sometimes omits to mention a fourth pulse, a local oscillator, that induces light emission at the time $t$ after the start of the experiment when the system was in its ground electronic state. The time $t$ is the time along which the dynamics evolve. The experiment is often discussed in terms of the time intervals between the laser pulses as shown in figure 1, so that the time $t$ is given by $t = T_0 + T_1 + T_2 + T_3$. 
In the simple case of a single monoexciton band and one photon transitions, the observed polarization is the emission to the ground state. It is given as the expectation value over the density matrix at the time $t$ of the transition dipole moment $\hat{\mu}$ between the $N$ excited states and the ground state: the dipole operator is the operator that induce emission only from the $N$ excited states to the GS and is therefore of the form

$$\hat{\mu} = \sum_{m=1}^{N} \mu_{GS-m} |GS\rangle \langle m|$$

(0)

This is because in the set-up as discussed, the carrier frequency of the pulses is roughly one photon resonant with the monoexcitons and the duration of the pulse is such that several monoexcitons fall within its coherent energy bandwidth.

The density matrix up to time $t$ is:

$$\hat{\rho}(t) = \sum_{k,l} \hat{\rho}_{kl}(t) |k\rangle \langle l|$$

(0)

Only a small submatrix of this $\hat{\rho}(t)$ is probed by the emission to the ground state. It is the submatrix where the $N$ excited states, $l=1,\ldots, N$ emit to the ground state, $k=GS$ . Note an essential fact, not explicit in the notation in equation (0), that the density matrix implicitly depends on the coherences between the excited states. This will be clear when one explicitly shows the dependence on the time interval $T_2$.

For any given triplet of time intervals $T_1, T_2, T_3$ one can express the polarization, as

$$P_{T_1,T_2,T_3}(t) = \text{Tr}[\hat{\mu} \hat{\rho}(t)] = \sum_{l=1}^{N} \rho_{GS-l}^{T_1,T_3}(t) \mu_{l-GS}$$

(0)

$P_{T_1,T_2,T_3}(t)$ is a complex number and the emitted field is proportional to $iP_{T_1,T_2,T_3}(t)$. In 2DES, the polarization signal at a given time interval $T_2$ is conveniently represented as a 2D discrete map namely a matrix where the axis or equivalently the matrix elements are indexed by the intervals $T_1$ and $T_3$. One can then assemble the matrices for different values of $T_2$ into a tensor with elements indexed by the three time intervals, $T_1, T_2, T_3$, see Figure 1. Our notation above reflects the physical result that the density matrix depends not only on the actual value of the time $t$ but also on how the sum $t$ is made up in terms of the triplet of times, $T_1, T_2, T_3$. 
To interpret a pixel of a 2D map at a particular triplet \( T_1, T_2, T_3 \) it is often convenient to use third order perturbation theory.\(^{(1, 32)}\) Each matrix, which is at a given value of \( T_2 \), is typically plotted after a Fourier transform along the \( T_1 \) and \( T_3 \) axes and is thereby a matrix indexed by the frequencies \( \omega_1 \) and \( \omega_3 \). In the ideal case of well-resolved excited states the peaks are at the frequencies \( \omega_1 \) and \( \omega_3 \) that are the energies of the excited states in atomic units \( \hbar = 1 \). As we vary \( T_2 \) each peak in the ideal case has the phase

\[
\vartheta_{ij} = \exp\left( (\omega_{1i} - \omega_{3j})t \right)
\]

(0)
corresponding to the coherence between the two excited states \( i \) and \( j \) at a particular time \( t \). The energies of the dots depend on their size so when there is a not negligible size disorder there can be coherences that beat in more than one given pixel on the frequency map. The number of excited states that contribute to the response, \( N \), can therefore be smaller than the number of pixels, \( N_p \) (which is determined by the sampling along \( T_1 \) and \( T_3 \)). The analysis of the beating periods along \( T_2 \) can be done by performing a third FT along \( T_2 \).\(^{(12, 35)}\) This is not the most stable method and one can possibly get artifacts.

**Intermezzo: A simple example of quantum computing with the hardware**

We aim for diverse applications of the hardware towards quantum computing. Here is a preliminary report of a very simple but quite explicit application to the quantum vibrational dynamics of a linear triatomic molecule. This extends our earlier work\(^{(33, 34)}\) in that we do not need the vibrations to be harmonic although it still simplifies the procedure when they are. The two vibrational coordinates are described on a discrete square grid with small spacings along the bond distances. A map of the amplitude of the wave function of the molecule on different grid points at a given time is mathematically equivalent to a spectral map of a 2D spectrum at a given time \( T_2 \). Instead of using localized bases function as needed for a DVR \(^{(36)}\) one can expand the wave function of the molecule in other basis states, e.g., eigenfunctions of a useful zeroth order Hamiltonian. Here too, for two coupled vibrations a map of the amplitude of the wave function on different basis functions at a given time is mathematically equivalent to a spectral map of a 2D spectrum at a given time \( T_2 \).

To simulate time change we start with an initial non stationary state of the molecule that is represented by the spectral map at a time shortly after the second pulse is over. The subsequent
dynamics occur along the $T_2$ time axis. We distinguish between harmonic and anharmonic vibrations. In the harmonic case there are three frequencies and typically these can be matched or nearly so with frequencies of coherences in the hardware. So the evolution along $T_2$ is readily accounted for as shown in our earlier work.(33, 34) Otherwise, and for anharmonic vibrations, we need to stretch or squeeze segments along the axis of $T_2$ so the phase change of the time evolving vibrations matches the phase changes of the coherences due to the different spacings provided by the electronic excitonic manifold of the hardware. This phase matching is covered in more detail by the complementary analysis using the variable projection approach.(37) The SVD procedure can be extended to more than two dimensions(38) and so the dynamics of more coupled vibrations can be emulated.

**Characterizing the spectral maps of the hardware**

We use the well documented singular value decomposition, SVD, as a way to characterize the entire spectral output of the hardware in one expression. To do so we first generate a matrix $A$ where each row is a vector labelled by a particular time interval $T_2$. The $N_p = N_{T_1} \times N_{T_3}$ elements of the vector are the observed intensities at each pixel such that the row is a vectorized form of the 2D spectral matrix at that time $T_2$. $N_{T_1}$ and $N_{T_3}$ are the number of times sampled along $T_1$ and $T_3$ respectively. The matrix $A$ is rectangular with dimensions $N_{T_2} \times N_p$, where $N_{T_2}$ is the number of points samples along $T_2$. One needs to sample $T_2$ to long times to uncover all the periods of the coherences between excited states. So in general $N^2 < N_p$ and we take it that $N_p > N_{T_2}$. The SVDecomposition of $A$ is the matrix product $A = USV^\dagger$ or, explicitly, for the matrix element at the $i$'th row, a pixel index, meaning that $i$ is the double index $\omega_1, \omega_3$ and the column, a time $t_k$ along $T_2$,

$$
(A)_{it_k} = \sum_{\alpha=1}^{M} U_{i\alpha} S_{\alpha} V_{t_k\alpha}^* = \sum_{\alpha=1}^{M} \lambda_{\alpha t_k} U_{i\alpha} 
$$

(0)

Here $S_{\alpha}$ is a singular value, an element of the diagonal matrix $S$ and in the notation of surprisal analysis(39) $\lambda_{\alpha t_k} = S_{\alpha} V_{t_k\alpha}^*$ is a time dependent multiplier, the weight of the component $\alpha$ at the
time $t_k \cdot U_{i\alpha}$ is a time independent element of the constraint $\alpha$ giving the weight of pixel $i$ in that constraint. $M$ is the number of terms in the SVD decomposition. To get an exact representation, one needs to include $M = N_{T_2}$ terms. We rank the terms in the sum in decreasing order of the singular values $S_\alpha$. The inevitable experimental noise means that the constraints with lower values of the $S_\alpha$ s are the more corrupted by noise and so the sum is often truncated keeping far fewer terms than $N_{T_2}$ and thereby providing a very compact representation for the matrix $A$ which is the matrix representing the entire output of the hardware.

To keep only the number of eigenvalues above noise, Shrager(40) suggests computing the autocorrelation function of the normalized eigenvectors $V$ and $U$.

**Example: a SVD decomposition of a model hardware.**

We model an ensemble of dimers of small CdSe dots with 5% disorder in size of the dots. The details of the electronic states of the dimer are given in much detail in our two earlier papers(33, 34) and in their supplementray information files in particular. The essence is that the 24 states in the first excited, monoeitonic band of the monomer are clustered in four bands of states as shown in figure 2. In the dimer the four monomer bands couple to eight bands as shown. The 2D spectrum is generated as described(34) with $N_{T_1} = N_{T_3} = 200$ points for $\omega_1$ and for $\omega_3$ generating $N_p = 40,000$ pixels. The spectrum is that of an ensemble of dimers and because of the finite size dispersion each coherence address in the $(\omega_3, \omega_1)$ plane is broadened as described(13) The SVD analysis reported below is done on the real part of the maps. We find 10 significant singular values $S_\alpha$ that come roughly by pairs, see figure 2.
The largest component shows the most intense diagonal population peak (the brightest $S_1$ band of the dimer, it is the highest one, the lowest one should be dark if we had no size dispersion). The 4 next eigenvalues correspond to coherences between states of the $S_1$ and $S_2$ bands. The next 2 correspond to the coherences between the two bands of the split $S_1$ band of the dimer (the highest one is brighter than the lowest one), and we have a longer period $\approx 30$ fs. All together SVD provides important 4 pairs of coherences in addition of the two largest singular values, that are population peaks, to get a good description of the input maps.

### The learning machine

The SVD derived eigenvector $U_{i\alpha}$ is the weight relation between the visible pixel $i$ to the invisible component $\alpha$. The importance of this connection at the time $t_k$ is given by the time dependent weight $\lambda_{i\alpha t_k} = S_\alpha V_{i\alpha t_k}^\star$. The singular value $S_\alpha$ determines the overall importance of the component while $V_{i\alpha t_k}$ is fractional importance at the time $t_k$. The $M$ invisible components identified by the SVD generate, equation (0), $$(A)_{it_k} = \sum_{\alpha=1}^{M} U_{i\alpha} \lambda_{\alpha t_k}$$ reproduce the readout at pixel $i$ at that time.

The steep drop of the singular values for $\alpha > M$, figure 2, means that the reproduction with $M =$
10 is accurate for our hardware model. Keeping all the $N_{T_2}$ singular values provides a mathematically exact reproduction.

**The initial state**

For the purpose of computing, the input to the machine is provided by the first two lasers. The initial state is the map $A$ at a very short time after $T_2 = 0$ just so that the second short laser pulse is already negligible. Along a frequency axis of the map there are more pixels than quantum states, $N_{T_1}, N_{T_3} > N$. The logic of the machine is based on the dynamics of $N$ quantum states. Therefore the machine can learn more than one initial state at a time.
Dynamics of a coherence $\langle \hat{E}_\alpha \rangle$ as a function of time along the $T_2$ axis

We aim to discuss the coherences and their evolution along the $T_2$ axis for the general case of an $N$ level system but with a Hamiltonian $H$ that is not necessarily the Hamiltonian of the unperturbed hardware. If $H$ is the Hamiltonian of the unperturbed hardware then each coherence oscillates along $T_2$ with its own frequency, equation (0) for example. We discuss the more general case because we aim to examine the result how the coherences of the hardware behave when the Hamiltonian of interest is that of the system we wish to emulate. We will show that we can relate the two so that in principle we show how to transform the dynamics of the system we need to emulate to the model spectral matrix of the hardware.

The first restriction that we impose is that the system to be emulated is described realistically in an $N$ dimensional Hilbert space, just like the hardware. Of course, using just $N$ electronic excited states of the hardware and forgetting for example the phonons means that we provide a description of the hardware valid only for a short time interval after excitation. But all that we need is a description valid before dephasing seriously sets in and it is typically the size disorder that limits this time range. For the system it may be that $N$ states are all that we really need, see (33, 34) for example. In a Lie algebraic approach it is sufficient when we identify $N^2$ (or fewer) generators that are closed under commutation with the Hamiltonian.

To a Hamiltonian there corresponds a Liouvillian, $L$, a ‘super operator’ that acts on operators as $[H, \hat{\mathcal{E}}]$ thereby generating the motion in time in the Heisenberg picture. Writing $\exp(iLt)$ as the form of the evolution super operator acting on the $N \equiv N^2$ operators we have that the coherences in any orthonormal basis $i,j$ at a time $t_k$ along $T_2$ is given as

$$E_{ij}(t) = \sum_{kl=1}^{N} \left( \exp(iLt_k) \right)_{j,k} E_{kl}(0) \quad (0)$$

The Liouvillian $L$ in equation (0) corresponds to the Hamiltonian of the hardware. Along $T_2$ this Liouvillian is a diagonal operator on the coherences of the hardware. Explicitly, starting with the $N$ eigenstates of this Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{N} \left| \alpha \right\rangle e_{\alpha} \left\langle \alpha \right| \quad (0)$$

where the $N = N^2$ coherence operators are $\hat{E}_{\alpha\beta} = \left| \alpha \right\rangle \left\langle \beta \right|$ so that, in the eigenstates basis,

$$\hat{L} \hat{E}_{\alpha\beta} = \left( e_{\alpha} - e_{\beta} \right) \hat{E}_{\alpha\beta} \quad (0)$$
Computing by observables

We take it that the same set of observables \( \{X_i\} \) is used to describe the hardware and the target system that we wish to emulate. Typically the \( X_i \)'s are coherences so that \( i \) is a double index, say \( \alpha, \beta \). The observables are assumed to close a Lie algebra

\[
\left[ \hat{X}_j, \hat{X}_i \right] = \sum_k^N C_{ji}^k \hat{X}_k
\]  

(1)

This is inherently satisfied when the observables are coherences, \( X_i = E_{\alpha \beta} \).

The Hamiltonian of the target system that we wish to emulate is taken to be a linear combination of the observables. This is inherently satisfied when we work in a finite dimensional Hilbert space

\[
H = \sum_i^N h_i X_i
\]  

(1)

The equations of motion of the observables are therefore closed

\[
\frac{i\hbar}{dt} \frac{d\hat{X}_i}{dt} = \left[ \hat{X}_j, \hat{H} \right] = \left[ \hat{X}_j, \sum_i h_i \hat{X}_i \right] = \sum_i h_i \sum_k C_{ji}^k \hat{X}_k = \sum_k \hat{X}_k \sum_i C_{ji}^k h_i
\]  

(1)

Only the possibly time dependent coefficient \( \{h_i\} \) is special to the target system that we wish to emulate. The structure constants \( C_{ji}^k \) are common to the hardware and the target system. Taking the expectation value of both sides on the density matrix, cf. Equation (2), leads to a closed set of linear first order equations for the mean values of the observables

\[
i\hbar \left( d\left\langle X_j \right\rangle / dt \right) = i\hbar \left( dTr(\rho X_j) / dt \right) = \sum_k L_{jk} \left\langle X_k \right\rangle
\]  

(1)

where \( L_{jk} = \sum_i C_{ji}^k h_i \) and our aim is to solve for the mean values for the set of given \( \{h_i\} \)'s.

The equations of motion for the hardware are

\[
\frac{i\hbar}{dt} \frac{d\hat{X}_i}{dt} = \left[ \hat{X}_j, \hat{H}' \right] = \left[ \hat{X}_j, \sum_i h'_i \hat{X}_i \right] = \sum_i h'_i \sum_k C_{ji}^k \hat{X}_k = \sum_k \hat{X}_k \sum_i C_{ji}^k h'_i
\]  

(1)

and the solution for the \( \left\langle X_j \right\rangle \)'s as a function of time is directly read on the hardware.

whose solution we want to determine for a known set \( \{h_i\} \) of coefficients.
The Lie-algebraic approach

For the fundamental algebraic approach we follow the work of Wei and Norman (41, 42) and its more explicit presentations, e.g., (43). The problem we wish to solve is a realization of what Feynman (30) described as an emulation: using the dynamics of one system to mimic the dynamics of another system that we call the target. We make the key assumption that the target can be described in the same $N$ dimensional Hilbert space that is used for the hardware. We take $N$ to be as large as practical for the hardware and so the Hilbert space is large. When the Hamiltonian matrix is $N$ by $N$ dimensional the corresponding Liouville matrix is $N^2$ by $N^2$ dimensional. The notation is simplified by the definition $\mathbb{N} \equiv N^2$ as above. The Hamiltonian for the target is written as $\hat{H} = \sum_{i=1}^{\mathbb{N}} \hat{h}_i \hat{X}_i$ with the key technical point is that it is the same set of observables as for the hardware. To describe the dynamics we follow Wei and Norman (41, 42) and factor the evolution operator in a product form, which yields for the propagation of the observables in time in terms of a set of $\mathbb{N}$ time dependent complex coefficients $\{\xi_{kl}\}$'s

$$\hat{X}_i(t) \equiv U(t)\hat{X}_i U(t)^\dagger = [\prod_{i=1}^{\mathbb{N}} \exp\left(g_i(t)\hat{X}_i\right)]\hat{X}_i[\prod_{i=1}^{\mathbb{N}} \exp\left(-g_i(t)\hat{X}_i\right)] = \sum_{k=1}^{\mathbb{N}} \xi_{kl}(t)\hat{X}_k \quad (1)$$

This Lie algebraic relation is at the heart of surprisal analysis (44) and next the dynamical correlation matrix $\xi$ is exhibited in its explicit detail with full details of the derivation in the SI. Note that the $\xi$ matrix is $N^2$ by $N^2$ dimensional. It replaces solving numerically the Heisenberg equations of motion for the observables. One can show that the coefficients $\{\xi_{kl}\}$ are analytic functions of the coefficients $\{g_i(t)\}$. A model analytic computation of the coefficients $\{\xi_{kl}\}$ for a quantum dot with two excited levels is presented next in some detail. For this model $N=3$ being a ground and two excited levels so that there are 81 time correlation coefficients $\{\xi_{kl}\}$. The matrix can be quite sparse so that there are fewer than $N^4$ non zero matrix elements. Still, this high dimensional matrix highlights the quantum advantage of our scheme of computing by $(N^2)$ observables.
An explicit Lie-algebraic solution for a two excited levels model

The three level \( (N=3) \) system, a ground state and two excited states, with the laser width spanning the energies of the two excited levels, as shown schematically in Fig. 3

![Figure 3: Scheme of the excitation level for a system with two excited states.](image)

There are nine \( E_{ij} \) operators but only 8 are linearly independent because of constrained by normalization, \( \hat{I} = \sum_{i,j} \hat{E}_{ij} \). The Hamiltonian of the hardware is

\[
\hat{H}'(t) = \hat{H}_0 - E(t) \hat{\mu} = \alpha_2 \hat{E}_{22} + \alpha_3 \hat{E}_{33} - E(t) \mu_{12} (\hat{E}_{12} + \hat{E}_{21}) - E(t) \mu_{13} (\hat{E}_{13} + \hat{E}_{31})
\]  

(1)

\( \hat{H}_0 \) is diagonal with \( \alpha_2 \) and \( \alpha_3 \) as the two excited state energies. In a somewhat sparse 3 by 3 matrix form:

\[
\hat{H}'(t) = \begin{pmatrix}
0 & -E(t)\mu_{12} & -E(t)\mu_{13} \\
-E(t)\mu_{12} & \alpha_2 & 0 \\
-E(t)\mu_{13} & 0 & \alpha_3
\end{pmatrix}
\]  

(1)

The commutation relations, written in the algebraic ‘ad’ notation are

\[
ad_{ij} \hat{E}_{kl} \equiv \left[ \hat{E}_{ij}, \hat{E}_{kl} \right] = \hat{E}_{il} \delta_{jk} - \hat{E}_{kj} \delta_{li}
\]

\[
ad_{ij}^2 \hat{E}_{kl} = \left[ \hat{E}_{ij}, \left[ \hat{E}_{ij}, \hat{E}_{kl} \right] \right] \ldots \text{etc}
\]  

(1)

For example

\[
ad_{ik}^2 \hat{E}_{il} = 0, \quad ad_{ij} \hat{E}_{kj} = 0 \ldots
\]  

(1)

In the commutators that are not zero, we have

\[
ad_{ij} \hat{E}_{ij} = \hat{E}_{ii} - \hat{E}_{jj}, \quad ad_{ij}^2 \hat{E}_{ij} = ad_{ij} \hat{E}_{ii} - ad_{ij} \hat{E}_{jj} = -2\hat{E}_{ij}, \quad ad_{ij}^3 \hat{E}_{ji} = -\left[ \hat{E}_{ij}, \hat{E}_{ij} \right] = 0
\]  

(1)

Using
\[ e^{g_j \hat{X}_j \hat{X}_i} = e^{g_j \hat{X}_j} e^{-g_j \hat{X}_j} = \sum_{l=0}^{\infty} \left( \frac{g_j}{l!} \right)^l \hat{X}_j^l \hat{X}_i \]

(1)

We have four cases:

(i) From Eq. (1)

\[ \exp \left( g_m A_{E_j} \right) E_{ik} = E_{ik} + g_m \left[ E_{ii}, E_{ij} \right] + 0 = E_{ik} + g_m E_{il} \]

(1)

(ii) From Eq. (1)

\[ \exp \left( g_m A_{E_i} \right) E_{ji} = E_{ji} + g_m \left[ E_{ji}, E_{ji} \right] + \frac{g_m^2}{2} \left( [E_{ij}, E_{ij}] - [E_{ij}, E_{ij}] \right) = E_{ji} + g_m (E_{ji} - E_{ji}) - g_m^2 E_{ij} \]

(1)

(iii) \[ [E_{ii}, E_{ij}] = E_{ij} \]

\[ \exp \left( g_1 a_{E_{i11}} \right) \hat{E}_{12} = \hat{E}_{12} = g_1 E_{12} + g_1^2 E_{12} + ... = \exp (g_1) \hat{E}_{12} \]

(1)

(iv) \[ [E_{ii}, E_{ji}] = -E_{ji} \]

\[ \exp \left( g_1 a_{E_{i11}} \right) \hat{E}_{ji} = E_{ji} + g_1 (E_{ji}) + \frac{g_1^2}{2} \left[ E_{ji}, E_{ji} \right] + \frac{g_1^3}{3!} \left[ E_{ji}, E_{ji} \right] + ... = \exp (-g_1) E_{ji} \]

(1)

Additional examples are computed in the same manner.

The elements of the matrix \( \mathbf{\xi} \) are defined in equation (1). So, for example

\[ \exp \left( g_1 a_{\hat{X}_1} \right) \hat{X}_2 = \exp \left( g_1 a_{\hat{E}_{11}} \right) \hat{E}_{12} = g_1 E_{12} + g_1^2 E_{12} + ... = \exp (g_1) \hat{E}_{12} = \exp (g_1) \hat{X}_2 \]

(1)

or \( \xi_{22} = \exp (g_1) \). The full details of the computations are in the appendix. The resulting sparse matrix \( \mathbf{\xi} \) is given below
\[
\begin{align*}
1 & 0 & 0 & g_2 & -g_2 g_4 & 0 & g_3 + g_6 e^{g_1} g_2 & e^{g_1} g_3 g_4 - g_6 g_4 g_2 & -g_7 g_3 - g_7 g_6 e^{g_1} g_2 + g_8 e^{-g_1} g_4 g_3 + g_8 g_6 g_4 g_2 \\
0 & e^{g_1} & 0 & -e^{g_1} g_2 & -e^{g_1} g_2^2 & e^{g_1} g_2 g_4 & -g_3 g_2 e^{g_1} - g_6 e^{g_1} g_2^2 e^{g_1} & e^{-g_1} g_5 e^{g_1} (1 + g_4 g_2) & g_7 g_3 g_2 e^{g_1} + g_7 g_6 e^{g_1} g_2^2 e^{g_1} + g_8 e^{-g_1} g_3 (1 + g_4 g_2) \\
0 & 0 & e^{g_1} & -e^{g_1} g_2 g_3 & e^{g_1} g_2 g_4 & e^{g_1} g_4 & -g_6 e^{g_1} g_3 g_2 e^{g_1} & e^{-g_1} g_4 g_3 e^{g_1} + g_6 e^{g_1} g_2 g_3 e^{g_1} + g_7 g_3 e^{g_1} + g_7 g_6 e^{g_1} g_3 g_2 e^{g_1} + g_8 g_6 e^{g_1} e^{g_1} g_2 e^{g_1} \\
0 & 0 & 0 & e^{-g_1} & -e^{-g_1} g_4 & 0 & g_6 e^{g_1} e^{-g_1} & -g_6 e^{g_1} e^{-g_1} & g_7 g_6 e^{g_1} e^{-g_1} + g_6 g_4 e^{g_1} e^{-g_1} \\
0 & 0 & 0 & -g_2 & (e^{g_2} + g_2 g_4) & 0 & -g_6 e^{g_1} g_2 & g_6 e^{g_1} g_2 + g_6 & g_7 g_6 e^{g_1} g_2 - g_8 g_6 - g_8 g_6 g_4 g_2 \\
0 & 0 & 0 & -g_3 & g_3 g_4 & e^{g_1} & -g_6 e^{g_1} g_3 & g_6 e^{g_1} g_3 - g_6 e^{g_1} g_3 & g_7 g_6 e^{g_1} + g_7 g_6 e^{g_1} g_3 + g_8 g_6 e^{g_1} g_3 + g_8 g_6 e^{g_1} e^{g_1} g_3 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{-g_1} & -e^{-g_1} g_4 e^{-g_1} & g_7 e^{-g_1} + g_7 g_4 e^{-g_1} e^{-g_1} \\
0 & 0 & 0 & 0 & 0 & 0 & -g_2 & e^{-g_1} (1 + g_4 g_2) & -g_8 e^{-g_1} + g_8 e^{-g_1} g_4 g_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -g_3 & -g_6 + e^{-g_1} g_4 g_3 & 1 + g_7 g_3
\end{align*}
\]
In terms of the sparse time correlation matrix $\xi$ we have the explicit expression for the time evolution of the observables, cf. equation (1)

$$\hat{X}_i(t) = \hat{U}(t) \hat{X}_i \hat{U}^+(t) = \sum_{k=1}^{N} \xi_{ki}(t) \hat{X}_k$$  \hspace{1cm} (2)

Furthermore, Wei and Norman show that the relation between the coefficients $\{g_{ij}\}$ that appear in the factorized evolution operator, cf. equation (2)

$$\hat{U} = \exp(g_{11} \hat{E}_{11}) \exp(g_{12} \hat{E}_{12}) \exp(g_{13} \hat{E}_{13}) \exp(g_{21} \hat{E}_{21}) \ldots \exp(g_{33} \hat{E}_{33})$$  \hspace{1cm} (2)

and the coefficients $\{h_j\}$ of the expansion of the Hamiltonian in the observables, equation (2), is

$$\begin{pmatrix}
  h_1(t) \\
  \vdots \\
  h_9(t)
\end{pmatrix} = 
\begin{pmatrix}
  \xi_{11}(t) & \cdots & \xi_{19}(t) \\
  \vdots & \ddots & \vdots \\
  \xi_{91}(t) & \cdots & \xi_{99}(t)
\end{pmatrix} 
\begin{pmatrix}
  g_1(t) \\
  \vdots \\
  g_9(t)
\end{pmatrix}$$  \hspace{1cm} (2)

Inverting this equation requires taking a smooth analytic continuation across any singular point of the inverse of the $\xi$ matrix, see, for example ref. (45).

**Concluding remarks**

Quantum processors are expected to build on the unique principles of quantum mechanics to make major improvements in the capabilities of information processing and computing. Here we proposed a design that is based on an array of quantum dots. The coherences between the states of the array are our logic variables. We described the dynamical response of such an array and furthermore ask if it can emulate the dynamics of other systems. The input is by nonlinear optical spectroscopy on an ensemble of dots and one reading of the device generates the entire output of interest. The output can be arranged as a matrix. Singular Value Decomposition, SVD, relates the output to a smaller set of hidden principal components. The output can be modified by changing the parameters of the spectroscopy that generates the input. A Lie algebraic discussion of the dynamics of the hardware of $N = 3$ levels is provided with complete explicit details of the $N = N^2 = 9$ coherences (=observables) and the resulting 9 by 9 sparse time correlation matrix $\xi$ for the dynamical algebra in a simple three levels model system. Already the simple model highlights the quantum advantage of our approach.
Acknowledgements

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Appendix: Details of the analytical computation of the 81 elements of the matrix $\xi$

We present, column by column the computations of the 81 elements of the matrix $\xi$ as given in equation (1) in the main text. The columns are labeled by $i=1,...,9$

**Column $i=1$**

$\xi_{11} = 1 , \xi_{k1} = 0$

**Column $i=2$**

$\exp(g_1 a d\hat{X}_1)\hat{X}_2 = \exp(g_1 a d\hat{E}_{11})\hat{E}_{12} = \hat{E}_{12} + g_1 E_{12} + \frac{g_1^2}{2} E_{12} + ... = \exp(g_1)\hat{E}_{12} = \exp(g_1)\hat{X}_2$

$\xi_{22} = \exp(g_1) , \xi_{k2} = 0$

**Column $i=3$**

$\exp(g_1 a d\hat{X}_1)\exp(g_2 a d\hat{X}_2)\hat{X}_3 = \exp(g_1 a d\hat{E}_{11})\exp(g_2 a d\hat{E}_{12})\hat{E}_{13} = \hat{E}_{13} + g_2 \left[ E_{12}, E_{13} \right] = \hat{E}_{13}$

$\exp(g_1 a d\hat{E}_{11})E_{13} = E_{13} + g_1 E_{13} + \frac{g_1^2}{2} E_{13} + ... = \exp(g_1)E_{13}$

$\xi_{33} = \exp(g_1) , \xi_{k3} = 0$ for all other $k$

**Column $i=4$**

$\exp(g_1 a d\hat{X}_1)\exp(g_2 a d\hat{X}_2)\exp(g_3 a d\hat{X}_3)\hat{X}_4$

$\exp(g_3 a d\hat{X}_3)\hat{X}_4 = \exp(g_3 a d\hat{E}_{21})\hat{E}_{22} = \hat{E}_{22} + g_3 \hat{E}_{22} + \frac{g_3^2}{2} \left[ E_{12}, E_{13} \right] = \hat{E}_{22} + g_3 \hat{E}_{23}$

$\exp(g_2 a d\hat{X}_2)\exp(g_3 a d\hat{X}_3)\hat{X}_4 = \exp(g_2 a d\hat{E}_{12})\left( E_{21} - g_3 \hat{E}_{23} \right)$

$= E_{21} + g_2 \left( E_{11} - E_{22} \right) - g_2^2 E_{12} + 0$

$\left( -g_3 \right) \left[ E_{23} + g_2 \hat{E}_{13} + \frac{g_2^2}{2} \left[ E_{12}, E_{13} \right] \right] = E_{21} + g_2 \left( E_{11} - E_{22} \right) - g_2^2 E_{12} - g_3 E_{23} - g_3 g_2 \hat{E}_{13}$

$\exp(g_1 a d\hat{E}_{11})\left( E_{21} + g_2 \left( E_{11} - E_{22} \right) - g_2^2 E_{12} - g_3 E_{23} - g_3 g_2 \hat{E}_{13} \right)$

$= e^{-g_1}E_{21} + g_2 \left( E_{11} - E_{22} \right) - g_2^2 e^{g_1} E_{12} - g_3 E_{23} - g_3 g_2 e^{g_1} E_{13}$
\[\exp(g_1A d_{X_1}) \exp(g_2A d_{X_2}) \exp(g_3A d_{X_3}) \exp(g_4A d_{X_4}) \hat{X}_5\]
\[= \exp(g_4a_4 \hat{X}_4) \hat{X}_5 = \exp(g_4a_4 \hat{E}_{21}) \hat{E}_{22} = E_{22} - g_4 \hat{E}_{21}\]
\[= \exp(g_5A d_{E_{21}})(E_{22} - g_4E_{21}) = E_{22} - g_4(E_{21} - g_3 \hat{E}_{23}) = E_{22} - g_4E_{21} - g_4g_3 \hat{E}_{23}\]
\[= \exp(g_2A d_{\hat{X}_2})(E_{22} - g_4E_{21} - g_4g_3 \hat{E}_{23})\]
\[= \exp(g_2a_2 \hat{E}_{12}) E_{22} - g_4 \exp(g_2a_2 \hat{E}_{12}) E_{21} + g_3g_4 \exp(g_2a_2 \hat{E}_{12}) \hat{E}_{23}\]
\[= e^{g_2} E_{22} - g_4(E_{21} + g_2(E_{11} - E_{22}) - g_2^2 E_{12}) + g_3g_4(E_{23} + g_2 \hat{E}_{13})\]
\[= e^{g_2} E_{22} - g_4 e^{-g_1} E_{21} - g_4 s_2 (E_{11} - E_{22}) + g_4 - g_4 g_2 e^{g_1} E_{12} + g_4 g_3 E_{23}\]
\[+ g_4 g_3 g_2 e^{g_1} E_{13}\]

**Column i=6**
\[\exp(g_1A d_{E_{11}}) \exp(g_2A d_{E_{12}}) \exp(g_3A d_{E_{13}}) \exp(g_4A d_{E_{21}}) \exp(g_5A d_{E_{22}}) \hat{E}_{23}\]
\[= \exp(g_5A d_{E_{22}}) \hat{E}_{23} = E_{23} + g_5 \hat{E}_{23} + \frac{g_5^2}{2} \hat{E}_{23} + ... = \exp(g_5) \hat{E}_{23}\]
\[\exp(g_4A d_{E_{21}}) e^{g_5} E_{23} = e^{g_5} E_{23}\]
\[\exp(g_3A d_{E_{13}}) e^{g_5} E_{23} = e^{g_5} E_{23}\]
\[\exp(g_2A d_{E_{12}}) e^{g_5} E_{23} = e^{g_5} (E_{23} + g_2 E_{13})\]
\[\exp(g_1A d_{E_{11}}) e^{g_5}(E_{23} + g_2 E_{13}) = e^{g_5}(E_{23} + g_2 e^{g_1} E_{13})\]

**Column i=7**
\[\exp(g_1A d_{E_{11}}) \exp(g_2A d_{E_{12}}) \exp(g_3A d_{E_{13}}) \exp(g_4A d_{E_{21}}) \exp(g_5A d_{E_{22}}) \exp(g_6A d_{E_{23}}) E_{31}\]
\[= \exp(g_6A d_{E_{23}}) \hat{E}_{31} = E_{31} + g_6 E_{21}\]
\[\exp(g_5A d_{E_{22}})(E_{31} + g_6 E_{21}) = E_{31} + g_6 e^{g_5} E_{21}\]
\[\exp(g_4A d_{E_{21}})(E_{31} + g_6 e^{g_5} E_{21}) = E_{31} + g_6 e^{g_5} E_{21}\]
\[\exp(g_3A d_{E_{13}})(E_{31} + g_6 e^{g_5} E_{21}) = E_{31} + g_3 E_{11} - g_3 E_{33} - g_3^2 E_{13} + g_6 e^{g_5} E_{21} - g_6 e^{g_5} g_3 E_{23}\]
\[
\exp\left(g_2 Ad_{E_{12}}\right)\left(E_{31} + g_3 E_{11} - g_3 E_{33} - g_3^2 E_{13} + g_6 e^{\xi_1} E_{21} - g_6 e^{\xi_1} g_3 E_{33}\right)
= E_{31} - g_2 E_{32} + g_3 E_{11} - g_3 E_{12} - g_3 E_{33} - g_3^2 E_{13} + g_6 e^{\xi_1} \left(E_{21} + g_2 (E_{11} - E_{22}) - g_2^2 E_{12}\right) - g_6 e^{\xi_1} g_3 E_{33} - g_6 e^{\xi_1} g_3 E_{13}
\]
\[
\exp\left(g_4 Ad_{E_{21}}\right)\left(E_{31} - g_2 E_{32} + g_3 E_{11} - g_3 E_{12} - g_3 E_{33} - g_3^2 E_{13} + g_6 e^{\xi_1} \left(E_{21} + g_2 (E_{11} - E_{22}) - g_2^2 E_{12}\right) - g_6 e^{\xi_1} g_3 E_{33} - g_6 e^{\xi_1} g_3 E_{13}\right)
\]
Column \(i=8\)
\[
\exp\left(g_1 Ad_{E_{13}}\right)\exp\left(g_2 Ad_{E_{12}}\right)\exp\left(g_3 Ad_{E_{13}}\right)\exp\left(g_4 Ad_{E_{21}}\right)\exp\left(g_5 Ad_{E_{22}}\right)
\]
\[
\exp\left(g_6 Ad_{E_{32}}\right)\exp\left(g_7 Ad_{E_{31}}\right) E_{32} = E_{32}
\]
\[
\exp\left(g_6 Ad_{E_{32}}\right) E_{32} = E_{32} + g_6 (E_{22} - E_{33}) - g_2^2 E_{23}
\]
\[
\exp\left(g_5 Ad_{E_{32}}\right)\left(E_{32} + g_6 (E_{22} - E_{33}) - g_2^2 E_{23}\right) = e^{-\xi_1} E_{32} + g_6 (E_{22} - E_{23}) - g_6 e^{\xi_1} E_{23}
\]
\[
\exp\left(g_4 Ad_{E_{31}}\right)\left(e^{-\xi_1} E_{32} + g_6 (E_{22} - E_{33}) - g_2^2 e^{\xi_1} E_{23}\right)
\]
\[
\exp\left(g_3 Ad_{E_{32}}\right)\left(e^{-\xi_1} E_{32} - e^{-\xi_1} g_4 E_{31} + g_6 (E_{22} - E_{33} - g_4 E_{21}) - g_6^2 e^{\xi_1} E_{23}\right)
\]
\[
\exp\left(g_2 Ad_{E_{31}}\right)\left(e^{-\xi_1} (E_{32} + g_3 E_{12}) - e^{-\xi_1} g_4 (E_{31} + g_3 (E_{11} - E_{33}) - g_3^2 E_{13})\right)
\]
\[
+ g_6 (E_{22} - E_{33}) - g_6 g_3 E_{13} - g_6 g_4 (E_{21} - g_3 E_{23}) - g_6^2 e^{\xi_1} E_{23}\right)
\]
\[
\exp\left(g_1 Ad_{E_{32}}\right)\left(e^{-\xi_1} E_{32} - e^{-\xi_1} g_4 E_{31} + g_6 (E_{22} - E_{33} - g_4 E_{21}) - g_6^2 e^{\xi_1} E_{23}\right)
\]
\[
\exp\left(g_2 Ad_{E_{31}}\right)\left(e^{-\xi_1} (E_{32} + g_3 E_{12}) - e^{-\xi_1} g_4 (E_{31} + g_3 (E_{11} - E_{33}) - g_3^2 E_{13})\right)
\]
\[
+ g_6 (E_{22} - E_{33}) - g_6 g_3 E_{13} - g_6 g_4 (E_{21} - g_3 E_{23}) - g_6^2 e^{\xi_1} E_{23}\right)
\]
\[
\exp\left(g_3 Ad_{E_{32}}\right)\left(e^{-\xi_1} E_{32} + g_6 E_{12} - e^{-\xi_1} g_4 (E_{31} - g_3 E_{32}) - e^{-\xi_1} g_4 g_3 (E_{11} - g_2 E_{12}) + e^{-\xi_1} g_4 g_3 E_{33}\right)
\]
\[
+ e^{-\xi_1} g_4 g_3^2 E_{13} + g_6 (E_{22} + g_3 E_{12}) - g_6 E_{33} - g_6 g_3 E_{13} - g_6 g_4 (E_{21} + g_2 E_{11} - g_2 E_{22} - g_2^2 E_{12})
\]
\[
+ g_6 g_3 g_4 (E_{23} + g_2 E_{13}) - g_6^2 e^{\xi_1} (E_{23} + g_2 E_{13})
\]
\[
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\]
\[
\exp(g_1 \alpha + g_2 \beta) (e^{-g_1} E_{32} + e^{-g_1} g_3 E_{12} - e^{-g_1} g_4 (E_{31} - g_2 E_{32}) - e^{-g_1} g_4 E_{11} - g_2 E_{12}) + e^{-g_1} g_4 E_{33}
\]
\[
+ e^{-g_1} g_4 g_5^2 E_{13} + g_6 (E_{32} + g_2 E_{12}) - g_6 E_{33} - g_6 g_3 E_{13} - g_6 g_4 (E_{21} + g_2 E_{11} - g_2^2 E_{12})
\]
\[
- g_6 g_4 g_5 (E_{32} + g_2 E_{13}) - g_6^2 e^{g_1} (E_{21} + g_2 E_{13})
\]
\[
= e^{-g_1} E_{32} + e^{-g_1} g_3 e^{g_1} E_{12} - e^{-g_1} g_4 e^{g_1} E_{31} + e^{-g_1} g_4 g_5 E_{32} - e^{-g_1} g_4 E_{11} + e^{-g_1} g_4 g_5^2 e^{g_1} E_{12}
\]
\[
+ e^{-g_1} g_4 g_5 E_{33} + e^{-g_1} g_4 g_5^2 e^{g_1} E_{13} + g_6 E_{22} + g_2 e^{g_1} E_{12} - g_6 E_{33} - g_6 e^{g_1} E_{13} - g_6 g_4 e^{-g_1} E_{21}
\]
\[
- g_6 g_4 g_5 E_{11} + g_6 g_4 g_5^2 E_{12} + g_6 g_4 g_5 E_{23} + g_6 g_4 g_5^2 E_{13} - g_6^2 e^{g_1} E_{23} + g_2^2 e^{g_1} E_{13}
\]

**Column 9**

\[
\exp(g_5 \alpha + g_6 \beta) E_{33} = E_{33} - g_8 E_{32}
\]
\[
\exp(g_7 \alpha + g_8 \beta) (E_{33} - g_8 E_{32}) = E_{33} - g_7 E_{31} - g_8 E_{32}
\]
\[
\exp(g_6 \alpha + g_7 \beta) (E_{33} - g_7 E_{31} - g_8 E_{32}) = E_{33} + g_6 E_{23} - g_7 (E_{31} + g_6 E_{21})
\]
\[
- g_8 (E_{32} + g_6 E_{22} - g_6 E_{33} - g_8 E_{23})
\]
\[
\exp(g_5 \alpha + g_6 \beta) (E_{33} + g_6 E_{23} - g_7 E_{31} + g_6 E_{21}) - g_8 (E_{32} + g_6 E_{22} - g_6 E_{33} - g_8 E_{23})
\]
\[
= E_{33} + g_6 e^{g_1} E_{23} - g_7 E_{31} - g_7 g_6 e^{g_1} E_{21} - g_8 e^{-g_1} E_{32} - g_8 g_6 E_{22} + g_8 g_6 E_{33} + g_8 g_6^2 e^{g_1} E_{33} + g_8 g_6^2 E_{23}
\]
\[
\exp(g_4 \alpha + g_5 \beta) (E_{33} + g_5 g_6^2 E_{23} - g_7 g_6^2 e^{g_1} E_{21} - g_8 g_6^2 e^{-g_1} E_{32} - g_8 g_6 E_{22} + g_8 g_6 E_{33} + g_8 g_6^2 e^{g_1} E_{33} + g_8 g_6^2 E_{23}
\]
\[
= E_{33} + g_6 g_5 g_6^2 E_{23} - g_7 g_5 g_6 E_{21} - g_8 g_5 g_6 e^{g_1} E_{32} - g_8 g_6^2 E_{22} + g_8 g_6 E_{33} + g_8 g_6^2 e^{g_1} E_{33} + g_8 g_6^2 E_{23}
\]
\[
\exp(g_3 \alpha + g_4 \beta) (E_{33} + g_4 g_5^2 E_{23} - g_7 g_5 E_{21} - g_8 g_5 g_6 E_{32} - g_8 g_6 E_{22} + g_8 g_6 E_{33} + g_8 g_6^2 E_{23})
\]
\[
= E_{33} + g_3 g_4 g_5^2 E_{23} - g_7 g_3 E_{21} - g_7 g_3 E_{31} + g_7 g_5 E_{33} + g_7 g_5^2 E_{23} - g_7 g_6 E^{g_1} (E_{21} - g_3 E_{23})
\]
\[
- g_8 e^{-g_1} (E_{32} + g_3 E_{12}) + g_8 e^{-g_1} g_4 E_{31} - g_8 e^{-g_1} g_4 E_{31} + g_8 e^{-g_1} g_4 E_{31} + g_8 e^{-g_1} g_4 g_3 E_{13} + g_8 e^{-g_1} g_4 g_3 E_{33} + g_8 e^{-g_1} g_4 g_3 E_{13}
\]
\[
- g_8 g_6 E_{22} + g_8 g_6 g_4 (E_{21} - g_3 E_{23}) + g_8 g_6 E_{33} + g_8 g_6 E_{33} + g_8 g_6 E_{33}
\]

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\[
\exp(g_2E_{12})(E_{33} + g_3E_{13} + g_6e^{g_1}E_{23} - g_7E_{31} - g_7g_3E_{11} + g_7g_3E_{33}) + g_7g_6g_1E_{23} - g_8e^{-g_1}(E_{32} + g_3E_{12}) + g_8e^{-g_1}g_4E_{31} - g_8e^{-g_1}g_4g_3E_{11} + g_8g_6g_3E_{23} + g_8g_6g_3E_{13} + g_8g_6e^{g_1}E_{23}
\]

\[
= E_{33} + g_3E_{13} + g_6e^{g_1}E_{23} + g_6e^{g_1}g_2E_{13} - g_7E_{31} - g_7g_3E_{11} + g_7g_3g_2E_{12} + g_7g_3E_{33} + g_7g_3^2E_{13}
\]

\[
- g_7g_6e^{g_1}E_{21} - g_7g_6e^{g_1}g_2E_{11} + g_7g_6e^{g_1}g_2E_{22} + g_7g_6e^{g_1}g_2E_{12} + g_7g_6e^{g_1}g_3E_{23}
\]

\[
+ g_7g_6e^{g_1}g_3g_2E_{13} - g_8e^{-g_1}E_{32} + g_8e^{-g_1}g_4E_{31} - g_8e^{-g_1}g_4g_2E_{12} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
+ g_8e^{-g_1}g_4g_3E_{23} + g_8e^{-g_1}g_4g_3g_2E_{13} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
= E_{33} + g_3E_{13} + g_6e^{g_1}E_{23} + g_6e^{g_1}g_2E_{13} - g_7E_{31} - g_7g_3E_{11} + g_7g_3E_{33} + g_7g_3^2E_{13}
\]

\[
- g_7g_6e^{g_1}E_{21} - g_7g_6e^{g_1}g_2E_{11} + g_7g_6e^{g_1}g_2E_{22} + g_7g_6e^{g_1}g_2E_{12} + g_7g_6e^{g_1}g_3E_{23}
\]

\[
+ g_7g_6e^{g_1}g_3g_2E_{13} - g_8e^{-g_1}E_{32} + g_8e^{-g_1}g_4E_{31} - g_8e^{-g_1}g_4g_2E_{12} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
- g_8e^{-g_1}g_4g_3E_{23} + g_8e^{-g_1}g_4g_3E_{12} + g_8e^{-g_1}g_4g_3E_{33} + g_8e^{-g_1}g_4g_3g_2E_{13} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
+ g_8e^{-g_1}g_4g_3E_{23} + g_8e^{-g_1}g_4g_3g_2E_{13} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
\exp(g_1Ad_{11})
\]

\[
(E_{33} + g_3E_{13} + g_6e^{g_1}E_{23} + g_6e^{g_1}g_2E_{13} - g_7E_{31} - g_7g_3E_{11} + g_7g_3E_{33} + g_7g_3^2E_{13}
\]

\[
- g_7g_6e^{g_1}E_{21} - g_7g_6e^{g_1}g_2E_{11} + g_7g_6e^{g_1}g_2E_{22} + g_7g_6e^{g_1}g_2E_{12} + g_7g_6e^{g_1}g_3E_{23}
\]

\[
+ g_7g_6e^{g_1}g_3g_2E_{13} - g_8e^{-g_1}E_{32} + g_8e^{-g_1}g_4E_{31} - g_8e^{-g_1}g_4g_2E_{12} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
- g_8e^{-g_1}g_4g_3E_{23} + g_8e^{-g_1}g_4g_3E_{12} + g_8e^{-g_1}g_4g_3E_{33} + g_8e^{-g_1}g_4g_3g_2E_{13} - g_8e^{-g_1}g_4g_3E_{11} + g_8e^{-g_1}g_4g_3g_2E_{12}
\]

\[
- g_8g_6g_2E_{12} + g_8g_6g_4E_{21} + g_8g_6g_4E_{21} - g_8g_6g_4g_2E_{11} - g_8g_6g_4g_2E_{22} - g_8g_6g_4E_{21}
\]

\[
- g_8g_6g_4g_2E_{13} + g_8g_6g_4g_2E_{23} + g_8g_6e^{g_1}g_2E_{13}
\]
References
43. C. Altafini (2002) Explicit Wei-Norman formulae for matrix Lie groups. in *Proc 41st IEEE Conf. on Decision and Control*, pp 2714-2719