# Path integral representation of the Dirac equation and integration order

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**Abstract** In this paper we ask if there is an interesting twist in mathematics of the Feynman propagator of the present day [1] formalism for the Dirac equation. The case studied is with only a potential function $V$. If there is no special integration order per step for e.g. $d\mathbf{x}^{(0)} = dx_1dx_2dx_3$ and $d\mathbf{p}^{(0)} = dp_1dp_2dp_3$, then a delta function occurs when $c\Delta t \approx 0$ and $-\infty < p_j < \infty$ for $j = 1, 2, 3$. The found geoemtry of propagation can be tested in solid state chemistry. A crystal structure where only a two dimensional propagation of quasi particles can be created must be attached to the final step of the path integral. Neutron emission giving chemical isotopes is briefly discussed.

## 1 Introduction

In the modern literature [1] the Feynman path integral [2], [4] for the Dirac equation is written such as presented below. Generally speaking, the form that is employed for the Dirac equation is [1, eq 1.2 page 64], $i\hbar\frac{\partial}{\partial t}u(t,\mathbf{x}) = \mathcal{H}u(t,\mathbf{x})$. The $u$ in [1] is an $N \times 1$ vector. Here we will take N=4. In [1], $\mathbf{x} = (x_1, x_2, \dots x_d)$. Here we will take $d = 3$. The $N = 4$ and $d = 3$ will allow insight into physical application. The Hamiltonian in $N = 4$ and $d = 3$ is (no summation convention)

$$\mathcal{H}(t,\mathbf{x}) = c\sum_{k=1}^{3}\hat{\alpha}_k\left(p_k - eA_k(t,\mathbf{x})\right) + V(t,\mathbf{x}) + \hat{\beta}mc^2 \tag{1}$$

The $\hat{\alpha}$ and $\hat{\beta}$ are constant Hermitian $N \times N = 4 \times 4$ Clifford/Dirac matrices with $\hat{\alpha}_k\hat{\beta} + \hat{\beta}\hat{\alpha}_k = 0$, $\hat{\alpha}_m\hat{\alpha}_k + \hat{\alpha}_k\hat{\alpha}_m = 2\delta_{k,m}\hat{1}$ and $\hat{\beta}^2 = \hat{\alpha}_k^2 = \hat{1}$, for $k = 1, 2, 3$.

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So

$$\hat{\alpha}_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \text{ and } \hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

Here 1 and 0 are $4 \times 4$ unity and zero matrices respectively. Sometimes they indicate $2 \times 2$ matrices and sometimes scalars. The use will be clear from the context. The $\sigma$'s are the $2 \times 2$ Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

The $\hbar$ is the Planck constant, the $c$ is the velocity of light in vacuum and $e$ is the charge of an electron. Further $(V, \mathbf{A})$ represents the electromagnetic field conditions. Let us look into problems where $\mathbf{A}$ is absent. Therefore,

$$\mathcal{H}(t, \mathbf{x}) = c \sum_{k=1}^{3} \hat{\alpha}_k p_k + V(t, \mathbf{x}) + \hat{\beta} mc^2 \tag{4}$$

The Lagrangian associated to (1) is $\mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{x}} - \mathcal{H}(t, \mathbf{x})$. Here, $\dot{\mathbf{x}}(t) = \partial_t \mathbf{x}(t) = \frac{d}{dt}\mathbf{x}(t)$, i.e. the time derivative of the $\mathbf{x}(t)$. The $\mathbf{p} \cdot \dot{\mathbf{x}}$ is the inner product of two vectors. It is multiplied by the $4 \times 4$ unity matrix. The Lagrangian then is

$$\mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) = \hat{1} \sum_{j=1}^{3} p_j \dot{x}_j(t) - \mathcal{H}(t, \mathbf{x}) \tag{5}$$

Note that here the $\mathcal{H}$ is a $4 \times 4$ matrix, the $\hat{1}$ is the $4 \times 4$ unity matrix and therefore the $\mathcal{L}$ is a $4 \times 4$ matrix.

### 1.1 Feynman propagator

The Feynman propagator operation contains $K_{D_\Delta}(t, t_{\text{ini}})$ [1]. Let $\tau_j \in \mathbb{R}$ and $j = 0, 1, 2, \dots \nu - 1, \nu$. The time division, i.e. the collection of time points, is denoted with $\Delta = \{\tau_j | j = 1, 2, \dots \nu - 1, \nu\}$. We have $\tau_0 = t_i = t_{\text{ini}}$ and $\tau_{\nu+1} = t = t_{\text{fin}}$. Therefore, if $f = u(t_i)$ with $f = (f_1, f_2, f_3, f_4)$ and each $f_j = f_j(x_1, x_2, x_3)$, with $j = 1, 2, 3, 4$. The time approximation propagator is then written like $K_{D_\Delta}(t, t_{\text{ini}})f$. We take $\Theta_\Delta$ the collection intervals $(\tau_{j-1}, \tau_j]$ with $j = 1, \dots \nu$. Let the $\mathbf{x} \in \mathbb{R}^3$ be fixed and denote points on the path from $\mathbf{x}^{(0)} \in \mathbb{R}^3$ to $\mathbf{x} \in \mathbb{R}^3$ with $\mathbf{x}^{(j)}$ and here $j = 1, \dots \nu - 1$. This results in the paths $\big(\Theta_\Delta, \mathbf{q}_\Delta(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots \mathbf{x}^{(\nu-1)}, \mathbf{x})\big)$. Note $\mathbf{x} = \mathbf{x}^{(\nu)}$.

Let us in general define the variable

$$\mathbf{q}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta) = \mathbf{y} + \frac{\theta - s}{t - s}(\mathbf{x} - \mathbf{y}) \tag{6}$$

and $s \leq \theta \leq t$. The dot notation is $\dot{\mathbf{q}}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta) = \frac{d}{d\theta}\mathbf{q}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta)$. For the variable $\mathbf{p}$ a similar paths can be defined; $\left(\Theta_\Delta, \mathbf{p}_\Delta(\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(\nu-1)}, \mathbf{p})\right)$. Note $\mathbf{p} = \mathbf{p}^{(\nu)}$. Like with $\mathbf{x}$ the employment of the symbolism will be convenient for the presentation. When necessary the index $\nu$ will be used.

In this way we are able to define the action in interval $\theta \in (s, t]$. Here $s$ and $t$ are symbolic for consequtive $\tau$. Hence, for ease of notation $\mathbf{p} = \mathbf{p}^{(0)}$

$$S(t, s, \mathbf{x}, \mathbf{y}, \mathbf{p}) = \int_s^t \mathcal{L}\left(\theta, \mathbf{q}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta), \dot{\mathbf{q}}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta), \mathbf{p}\right) d\theta \tag{7}$$

The propagator, where here again $t = t_{\text{fin}}$ is the endpoint in time corresponding with the end position, $\mathbf{x}$, is therefore written as ($d = 3$ spatial coordinates)

$$K_{D_\Delta}(t, t_{\text{ini}}) f(\mathbf{x}) = \int\int e^{*iS(t, \mathbf{q}_\Delta, \mathbf{p}_\Delta)} f\left(\mathbf{x}^{(0)}\right) \mathcal{D}_{\mathbf{x}_\Delta}\mathcal{D}_{\mathbf{p}_\Delta} \tag{8}$$

In this equation,

$$\begin{aligned} \mathcal{D}_{\mathbf{x}_\Delta} &= d\mathbf{x}^{(0)} d\mathbf{x}^{(1)}, \dots, d\mathbf{x}^{(\nu-1)} \\ \mathcal{D}_{\mathbf{p}_\Delta} &= (2\pi)^{-3\nu} d\mathbf{p}^{(0)} d\mathbf{p}^{(1)} \dots, d\mathbf{p}^{(\nu-1)} d\mathbf{p}^{(\nu)} \end{aligned} \tag{9}$$

With, $d\mathbf{x}^{(\ell)} = dx_1^{(\ell)} dx_2^{(\ell)} dx_3^{(\ell)}$, etcetera, $\ell = 0, 1, 2, \dots \nu{-}1$. The $f\left(\mathbf{x}^{(0)}\right)$ follows [6, pg 257 eq 10 or 11]. See also [1]. Note also that the $\mathbf{q}$ in the definition of $S$ is based on (6).

With $\mathbf{x}^{(\nu)} = \mathbf{x}_{\text{fin}} = \mathbf{x}$ and $\tau_{\nu+1} = t_{\text{fin}} = t$ and containing the initial $\mathbf{x}_{\text{ini}}$ the "starred" exponent in (8) indicates the product, while $\mathbf{p}^{(\nu)}$ is added to the integration to meet $\mathbf{x}_{\text{fin}}$

$$\begin{aligned} &\exp\left[*iS\left(t, \mathbf{q}_\Delta, \mathbf{p}_\Delta\right)\right] = \\ &\exp\left[i\int_{\tau_\nu}^{t} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_{\nu+1},\tau_\nu}_{\mathbf{x},\mathbf{x}^{(\nu-1)}}(\theta), \dot{\mathbf{q}}^{\tau_{\nu+1},\tau_\nu}_{\mathbf{x},\mathbf{x}^{(\nu-1)}}(\theta), \mathbf{p}^{(\nu)}\right) d\theta\right] \times \\ &\exp\left[i\int_{\tau_{\nu-1}}^{\tau_\nu} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_\nu,\tau_{\nu-1}}_{\mathbf{x}^{(\nu-1)},\mathbf{x}^{(\nu-2)}}(\theta), \dot{\mathbf{q}}^{\tau_\nu,\tau_{\nu-1}}_{\mathbf{x}^{(\nu-1)},\mathbf{x}^{(\nu-2)}}(\theta), \mathbf{p}^{(\nu-1)}\right) d\theta\right] \times \\ &\exp\left[i\int_{\tau_{\nu-2}}^{\tau_{\nu-1}} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_{\nu-1},\tau_{\nu-2}}_{\mathbf{x}^{(\nu-2)},\mathbf{x}^{(\nu-3)}}(\theta), \dot{\mathbf{q}}^{\tau_{\nu-1},\tau_{\nu-2}}_{\mathbf{x}^{(\nu-2)},\mathbf{x}^{(\nu-3)}}(\theta), \mathbf{p}^{(\nu-2)}\right) d\theta\right] \times \\ &\dots\exp\left[i\int_{\tau_1}^{\tau_2} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}^{(0)}}(\theta), \dot{\mathbf{q}}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}^{(0)}}(\theta), \mathbf{p}^{(1)}\right) d\theta\right] \times \\ &\exp\left[i\int_{\tau_0}^{\tau_1} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_1,\tau_0}_{\mathbf{x}^{(0)},\mathbf{x}_{\text{ini}}}(\theta), \dot{\mathbf{q}}^{\tau_1,\tau_0}_{\mathbf{x}^{(0)},\mathbf{x}_{\text{ini}}}(\theta), \mathbf{p}^{(0)}\right) d\theta\right] \end{aligned} \tag{10}$$

In [1] there is no explicit $\mathbf{x}_{\text{ini}}$ to do justice to the compact

$$K_{D_\Delta}(t, t_{\text{ini}}) f(\mathbf{x}) = \int\int_{\mathbf{x}_{\text{ini}}}^{\mathbf{x}_{\text{fin}}} \exp\left[*iS\left(t, \mathbf{q}_\Delta, \mathbf{p}_\Delta\right)\right] f\left(\mathbf{x}^{(0)}\right) \mathcal{D}_{\mathbf{x}_\Delta}\mathcal{D}_{\mathbf{p}_\Delta} \tag{11}$$

but there is a $\mathbf{x}_{\text{fin}}$. This inclusion of $\mathbf{x}_{\text{ini}}$ is comparable to the treatment of the Schrödinger equation path integral representation: [2, eq (2.1.82)] or [3, eq (50)]

## 2 The p integration of $\mathcal{L}\left(\theta, \mathbf{q}^{\tau_1,\tau_0}_{\mathbf{x}^{(0)},\mathbf{x}_{\text{ini}}}(\theta), \dot{\mathbf{q}}^{\tau_1,\tau_0}_{\mathbf{x}^{(0)},\mathbf{x}_{\text{ini}}}(\theta), \mathbf{p}^{(0)}\right)$

Note then that $\mathcal{D}_{\mathbf{p}_\Delta}$ contains $d\mathbf{p}^{(0)}$ which we denote for the ease of notation with $dp_1dp_2dp_3$. Further, $\mathcal{D}_{\mathbf{x}_\Delta}$ contains $d\mathbf{x}^{(0)}$. And this is denoted for ease of notation by $dy_1dy_2dy_3$. In effect we have now the $S$ of (7). Let's write $\mathbf{q}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta) = \mathbf{q}(\theta)$ and $\dot{\mathbf{q}}^{t,s}_{\mathbf{x},\mathbf{y}}(\theta) = \dot{\mathbf{q}}(\theta)$. Hence, the $p_1$ containing component of integral of $\mathcal{L}$ in $S$ of equation (7), note that $y_1$ equals $x_1^{(0)}$ for ease of writing.

$$S_{p_1} = \int_s^t (p_1\dot{q}_1(\theta) - c\hat{\alpha}_1 p_1)\, d\theta = p_1(y_1 - x_{1,\text{ini}}) - c(t-s)\hat{\alpha}_1 p_1 \tag{12}$$

Now the order of $dp_1dp_2dp_3$ and of $dy_1dy_2dy_3$ can be interchanged and $dp_1$ can be first before $dy_1$. Looking at (8) and (7) we can then have a kind of intrinsic Fourier integration that occurs in the path integral

$K_{D_\Delta}(t, t_{\text{ini}})f(\mathbf{x}) =$

$$\int \cdots \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-ic(t-s)\hat{\mathcal{A}}p_1\right] dp_1}_{\sim\mathcal{F}[1]} \hat{\mathcal{K}}f\left(\mathbf{x}^{(0)}\right)\mathcal{D}_{\mathbf{x}_\Delta}\frac{dp_2dp_3}{(2\pi)^2}\mathcal{D}^{1,\nu-1}_{\mathbf{p}_\Delta} \tag{13}$$

with, from the theory, viz.(10)

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(\nu-1)}, p_2, p_3, \mathbf{p}^{(1)} \ldots, \mathbf{p}^{(\nu-1)})$$

and with $\mathcal{D}^{1,\nu-1}_{\mathbf{p}_\Delta} = (2\pi)^{-3(\nu-1)}d\mathbf{p}^{(1)} \ldots d\mathbf{p}^{(\nu-1)}$ viz. (9) and $\hat{\mathcal{A}}$ is defined by

$$\hat{\mathcal{A}} = \hat{\alpha}_1 - \frac{\hat{1}}{c}\left(\frac{y_1 - x_{1,\text{ini}}}{t-s}\right) \tag{14}$$

And so we can have a definition,

$$K^{(p_1)}_{\mathcal{D}_\Delta}(\infty, c(t-s)) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-ic(t-s)\hat{\mathcal{A}}p_1\right] dp_1 \tag{15}$$

Looking at the $p_1$ integral, a Dirac delta function could arise from the Fourier integral (13). In the next section we will look into this possibility more closely.

### 2.1 Delta function designated in the underbrace of (14)

Let us write $c\Delta t$ for $c(t-s)$. Then write $\frac{\Delta x}{c\Delta t}$ for $\frac{y_1 - x_{1,\text{ini}}}{c(t-s)}$. Then let us observe firstly that from (14), i.e. $\hat{\mathcal{A}} = \hat{\alpha}_1 - \hat{1}\left(\frac{\Delta x}{c\Delta t}\right)$

$$\hat{\mathcal{A}}^{-1} = \frac{1}{1 - \frac{\Delta x^2}{c^2\Delta t^2}}\left[\hat{\alpha}_1 + \hat{1}\frac{\Delta x}{c\Delta t}\right] \tag{16}$$

For completeness $\hat{1}$ is the unit $4 \times 4$ matrix. Secondly we also have

$$\exp\left[-2\pi ic\Delta\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right] = \frac{i}{2\pi}\frac{\hat{\mathcal{A}}^{-1}}{c\Delta t}\frac{d}{dp_1}\exp\left[-2\pi ic\Delta t\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right] \quad (17)$$

Here the $\epsilon_N^2 p_1$ is already suppressed for the range $|p_1| < N$.

$$\epsilon_N^2 = \frac{2}{N^2}\log\left(1 + \frac{1}{2}N^{1-\zeta}\right) \quad (18)$$

for $0 < \zeta < 1$. But it must be noted that it is not suppressed for $\epsilon_N^2 p_1^2 = \epsilon_N^2 N^2$ in $\exp\left[-\epsilon_N^2 N^2/2\right]$. And,

$$\lim_{N\to\infty}\frac{2}{N}\log\left(1 + \frac{1}{2}N^{1-\zeta}\right) = (1-\zeta)\lim_{N\to\infty}\frac{N^{-\zeta}}{1 + \frac{1}{2}N^{1-\zeta}} = 0 \quad (19)$$

justifies suppression $\epsilon_N^2 p_1$.

If we in the following take $N$ which eventually goes to infinity we find, keeping the "=" for simplicity

$$K_{\mathcal{D}_\Delta}^{(p_1)}(N, c\Delta t) = \int_{-N}^{N}\exp\left[-2\pi ic\Delta t\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right]dp_1 = \quad (20)$$

$$= \frac{i}{2\pi}\frac{\hat{\mathcal{A}}^{-1}}{c\Delta t}\int_{-N}^{N}\frac{d}{dp_1}\exp\left[-2\pi ic\Delta t\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right]dp_1$$

Hence,

$$\int_{-N}^{N}\exp\left[-2\pi ic\Delta t\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right]dp_1 = \quad (21)$$

$$= \frac{1}{2\pi}\frac{\hat{\mathcal{A}}^{-1}}{c\Delta t}\sin\left[2\pi ic\Delta t\hat{\mathcal{A}}N\right]\exp\left[-\epsilon_N^2 N^2/2\right]$$

Then let us suppose that $c\Delta t$ goes to zero. Hence,

$$\frac{\hat{\mathcal{A}}^{-1}}{c\Delta t} = \frac{c\Delta t}{c^2\Delta t^2 - \Delta x^2}\left[\hat{\alpha}_1 + \hat{1}\frac{\Delta x}{c\Delta t}\right] \to \left(\frac{-1}{\Delta x^2}\right)\hat{1}\Delta x = -\frac{\hat{1}}{\Delta x} \quad (22)$$

In addition, because

$$c\Delta t\hat{\mathcal{A}} = \hat{\alpha}_1 c\Delta t - \hat{1}\Delta x \quad (23)$$

we find, because $\hat{1}^k = \hat{1},\ k = 1, 2\ldots$, that

$$\sin\left[2\pi ic\Delta t\hat{\mathcal{A}}N\right] \to -\sin\left(2\pi N\Delta x\right)\hat{1} \quad (24)$$

And so, with (22) and (24)

$$K_{\mathcal{D}_\Delta}^{(p_1)}(N, 0) = \lim_{c\Delta t\to 0}\int_{-N}^{N}\exp\left[-2\pi ic\Delta t\hat{\mathcal{A}}p_1 - \epsilon_N^2 p_1^2/2\right]dp_1 = \quad (25)$$

$$= \hat{1}\left(\frac{\sin\left(2\pi N\Delta x\right)}{\Delta x}\right)\exp\left[-\epsilon_N^2 N^2/2\right]$$

This has a delta function format, if $N\left(1 + \frac{1}{2}N^{1-\zeta}\right)^{-1} \to \infty$ for $N \to \infty$ and $0 < \zeta < 1$. This result extends to 3 dimensions [5].

*2.1.1 Other axes / dimensions*

Now do note we can under $c\Delta t$ to zero, also consider integration over $p_2$.

$$K^{(p_2)}_{\mathcal{D}_\Delta}(N, c\Delta t) = \int_{-N}^{N} \exp\left[-2\pi i c\Delta t \hat{\mathcal{A}}' p_2 - \epsilon_N^2 p_2^2/2\right] dp_2 \tag{26}$$

Like with the definition in (14)

$$\hat{\mathcal{A}}' = \hat{\alpha}_2 - \frac{\hat{1}}{c}\left(\frac{\Delta x_2}{\Delta t}\right) \tag{27}$$

Similar to the case $j = 1$ in section-2.1, for the axis $j = 3$ and the axis $j = 2$, the same result arises.

## 3 Spatial integration

Looking at (25) we can conclude a delta function for $N \to \infty$ in $\Delta x$. This gives the integration

$$\int_{-\infty}^{\infty} w(y_1, y_2, y_3)\delta\left(y_1 - x_{1,\mathrm{ini}}\right) dy_1 = w(x_{1,\mathrm{ini}}, y_2, y_3) \tag{28}$$

Here $w$ is a complex vector function based on $f(\mathbf{y})$ and the fact that $\mathbf{y} = \mathbf{x}^{(0)}$ also occurs in $\mathcal{L}\left(\theta, \mathbf{q}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}^{(0)}}(\theta), \dot{\mathbf{q}}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}^{(0)}}(\theta), \mathbf{p}^{(1)}\right)$ of (10). The startpoint is $\mathbf{x}_{\mathrm{ini}}$.

Subsequent integration over $\frac{dp_2}{2\pi}$ and $dy_2$ and $\frac{dp_3}{2\pi}$ and $dy_3$ leaves us with $f(\mathbf{x}_{\mathrm{ini}})$ and a transformed $\mathcal{L}\left(\theta, \mathbf{q}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}_{\mathrm{ini}}}(\theta), \dot{\mathbf{q}}^{\tau_2,\tau_1}_{\mathbf{x}^{(1)},\mathbf{x}_{\mathrm{ini}}}(\theta), \mathbf{p}^{(1)}\right)$ in equation (10). With the integration over $\mathbf{p}^{(1)}$ we will see a similar $\mathcal{F}[1]$ Fourier transform as in the underbrace of (13). Therefore we find,

$K_{D_\Delta}(t, t_{\mathrm{ini}})f(\mathbf{x}) =$

$$\int \exp\left[iS_\nu\right] \exp\left[iS_{\nu-1}\right] w^{(\nu-1)}\left(\mathbf{x}_{\mathrm{ini}}\right) d\mathbf{x}^{(\nu-1)} \frac{d\mathbf{p}^{(\nu-1)}}{(2\pi)^3}\frac{d\mathbf{p}^{(\nu)}}{(2\pi)^3} \tag{29}$$

With, like in (7)

$$S_{\nu-1} = \int_{\tau_{\nu-1}}^{\tau_\nu} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_\nu,\tau_{\nu-1}}_{\mathbf{x}^{(\nu-1)},\mathbf{x}_{\mathrm{ini}}}(\theta), \dot{\mathbf{q}}^{\tau_\nu,\tau_{\nu-1}}_{\mathbf{x}^{(\nu-1)},\mathbf{x}_{\mathrm{ini}}}(\theta), \mathbf{p}^{(\nu-1)}\right) d\theta \tag{30}$$

$$S_\nu = \int_{\tau_\nu}^{\tau_{\nu+1}} \mathcal{L}\left(\theta, \mathbf{q}^{\tau_{\nu+1},\tau_\nu}_{\mathbf{x}_{\mathrm{fin}},\mathbf{x}^{(\nu-1)}}(\theta), \dot{\mathbf{q}}^{\tau_{\nu+1},\tau_\nu}_{\mathbf{x}_{\mathrm{fin}},\mathbf{x}^{(\nu-1)}}(\theta), \mathbf{p}^{(\nu)}\right) d\theta$$

and $w^{(\nu-1)}\left(\mathbf{x}_{\mathrm{ini}}\right)$ derives from the repeated delta function integration with $\mathbf{x}_{\mathrm{ini}}$ as constant.

The equations (29) and (30) give rise to a delta function integration in the $\mathbf{x}^{(\nu-1)}$ integration. To illustrate this, for $x_1^{(\nu-1)}$ we have

$$K_{D_\Delta}(t,t_{\text{ini}})f(\mathbf{x}_{\text{fin}}) = w^{(\nu-1)}(\mathbf{x}_{\text{ini}})\times$$

$$\int_{-\infty}^{\infty} \delta\left(x_1^{(\nu-1)} - x_{1,\text{ini}}\right) \underbrace{\left(\frac{\sin\left[2\pi N\left(x_{1,\text{fin}} - x_1^{(\nu-1)}\right)\right]}{\Delta x^{(\nu-1)}}\right) e^{\frac{-\epsilon_N^2 N^2}{2}}}_{\text{from } p_1^{(\nu)} \text{integration}} dx_1^{(\nu-1)} \quad (31)$$

with $\Delta x^{(\nu-1)} = x_{1,\text{fin}} - x_1^{(\nu-1)}$. And so looking at (10), the integration over the first term of (10) will give zero when $\mathbf{x}_{\text{fin}} \neq \mathbf{x}_{\text{ini}}$.

## 4 Conclusion & discussion

With the present day [1] formalism for a path integral for the $\mathbf{A} = \mathbf{0}$ Dirac equation, we have added the information of the initial position, $\mathbf{x}_{\text{ini}}$. We believed that this element was missing from [1]. It appears as though the, for convenience repeated below, path integral representation

$$K_{D_\Delta}(t,t_{\text{ini}})f(\mathbf{x}) = \int\int_{\mathbf{x}_{\text{ini}}}^{\mathbf{x}_{\text{fin}}} \exp\left[*iS\left(t,\mathbf{q}_\Delta,\mathbf{p}_\Delta\right)\right] f\left(\mathbf{x}^{(0)}\right) \mathcal{D}_{\mathbf{x}_\Delta}\mathcal{D}_{\mathbf{p}_\Delta}$$

allows a vanishing Feynman propagator for $\mathbf{x}_{\text{fin}} \neq \mathbf{x}_{\text{ini}}$ when $c\Delta t \approx 0$ and when $\mathbf{p}^{(\ell)} \in (-\infty,\infty)^3$ for all $\ell = 0,1,2\ldots,\nu$. Note, $\exp[-\epsilon_N^2 N^2/2] = 1/\left(1+\frac{1}{2}N^{1-\zeta}\right)$ via (18). Moreover, $\lim_{\Delta x\to 0}\left(\frac{\sin(2\pi N\Delta x)}{\Delta x}\right) = 2\pi N$. Note also that, therefore, (25) has a delta function behavior for $N\to\infty$ and that the latter arises from $\mathbf{p}^{(\ell)} \in (-\infty,\infty)^3$. We can summarize:

- Either the propagator vanishing is a physical consequence of a mathematically correct path representation.
  - If, $(\forall_{j=1,2,3})\, x_{j,\text{fin}} \neq x_{j,\text{ini}}$, the Feynman propagator vanishes.
  - Note that for $(\exists!_{j=1,2,3})\, x_{j,\text{fin}} \neq x_{j,\text{ini}}$ then

$$\lim_{N\to\infty} \frac{N^2/(2\pi)^2}{\left(1+\frac{1}{2}N^{1-\zeta}\right)^3} = \begin{cases} 0, & \zeta < \frac{1}{3} \\ \infty, & \zeta > \frac{1}{3} \\ 8, & \zeta = \frac{1}{3} \end{cases} \quad (32)$$

    In other words, a one dimensional difference between initial and final position allows a nonzero Feynman propagator here. In a similar vein we can have 2 coordinates different between $\mathbf{x}_{\text{fin}}$ and $\mathbf{x}_{\text{ini}}$.
  - Only in the step that ends in $\mathbf{x}_{\text{fin}}$ there is no "integrating over" of a delta function. In that case we have (32).
- Or an alternative is that the initial four vector $f\left(\mathbf{x}^{(0)}\right)$ is replaced by e.g. $f\left(\mathbf{x}^{(0)},\mathbf{p}^{(0)}\right)$ to avoid a collapse to delta functions under the condition $c\Delta t \approx 0$ and $\mathbf{p}^{(\ell)} \in (-\infty,\infty)^3$ for all $\ell = 0,1,2\ldots,\nu$.

Geometrically it looks as though the presented path integral would be able to describe certain solid state quasi particle forms of Dirac equation, with obviously, modified $c$. The chemical creation of crystals where only two dimensional propagation is possible, looks like good testing ground for this idea.

### 4.1 Neutron Dirac equation

In the creation of chemical isotopes the process of neutron emission occurs. An interesting example is Helium-5. For the $V$ of equation (4) we can take [7]

$$V(t,\mathbf{x}) = \mu \begin{pmatrix} \sum_{j=1}^{3} \sigma_j H_j(t,\mathbf{x}) & 0 \\ 0 & \sum_{j=1}^{3} \sigma_j H_j(t,\mathbf{x}) \end{pmatrix} \tag{33}$$

The propagator, whch is equivalent to the wave function vector, in the present paper is only nonzero under the already mentioned geometric conditions.

## Declarations

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