Marrying Wright's Law to Thermodynamics for an Ideal Relative Final Cost-predicting Model of Carbon-fuel Substitution

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Abstract

The problem of assessing the cost of C-fuel substitution is approached by means of a general interpretation of Wright's law and the introduction of the concept of thermodynamic utility which derives from energy carrier specific energy and storage state conditions. Via the Bienaymé–Chebyshev inequality, the ideal final cost ratio is determined at three different probabilities each with regard to compressed hydrogen, liquid hydrogen and Li-ion technology. The 96 % probability values supposedly balance insight and interval size best: however, cost-parity is not a result in any case. This paper points out evidence for the thesis that the subject of C-fuel substitution is governed by an intrinsic thermodynamic causality beyond economic and human factors, ultimately defining hidden pre-existing ideal baseline thresholds to the achievable in reality.

1. Introduction

WRIGHT described in 1936 a regularity between per-unit cost and manufacturing volume,[1] based on his observations at U.S. aircraft manufacturer Curtiss-Wright: with each doubling of production numbers the per-unit labour cost receded by a constant proportion. This empiric law became known under the Learning Curve Effect respective the more catchy term “Wright's law”. Equation 1 shows its mathematics with P_x being the per-unit price at cumulative quantity x, P_1 the initial per-unit price and b an empiric factor, typically $0.75 < b < 0.90$ applies.

$$P_{\mathrm{x}} = P_1\, x^{\log_2(b)} \qquad \textbf{(1)}$$

Despite its empiric nature and simplicity, Wright's law delivers the best overall performance in a comparative statistical analysis of six technology-cost forecasting models published in 2013 by NAGY *et al*, even ahead of Moore's vaunted law.[3] The study comprises 62 different technologies and the effect is prevalent across the board, though at a variety in emphasis. Because of Wright's law empiric nature, adopting an “agnostic view regarding the causal mechanism” is not only admissible but expedient.[4] While NAGY *et al* acknowledge the evident difficulty of reducing the manufacture of a complex item such as a car or airplane into a single coefficient, nevertheless a regularity of the kind applies. It is interesting to ask how this might relate to the cost of energy carrier value chains: due to the complex fabric of technological and human factors, from mining/prime provision to political prerequisites, this might be rejected as impossible altogether. However, at a closer look it looms that this does not hold true because a) energy is provisioned/exploited by means of matter-related physicochemical processes, the very essence of tangibility and b) the final utility of the “manufactured” commodity, the energy amount for use, is quantifiable, too. Thus, the contrary applies, because both ends of an energy carrier value chain are by principle tangibles, the reduction into a single figure index is actually favoured: The only question is how that might be achieved.

2. Methodical Approach

First, quantity b in equation 1 represents a cost advantage factor whose deeper causality is not known: it might be better production processes,[2] a workforce acquiring skill or a higher being providing guidance from outer space. Wright's law offers no means whatsoever to differ between these options: it only tells about the materialization of the cost advantage effect with unit production numbers. *Second*, the boundary conditions are assumed as ideal, i.e. unlimited access to skilled labour and materials in a free market with demand always slightly trailing ahead of production. This stable market-commodity supply scenario is vital for a linear development which is important for general comparability. The scenario is devoid of political interference of any kind. *Third*, in such a scenario the difference in cost between two energy carrier value chains cannot be otherwise vindicated than by their respective final thermodynamic utilities, given by the amount of specific energy made available. *Fourth*, the specific energy of an energy carrier may be expressed as pressure p per medium density ρ (uncommon but valid). This allows reducing the specific energy of an energy carrier into a single dimensionless figure if related to the pressure and medium density in the storage state prior final utilization. That particular condition is emphasized because it represents the apex of the effort related to energy carrier provision: it has been produced or mined, purified and filled into the tank, awaiting its final destination. Furthermore, this approach makes it possible to compare e.g. gaseous energy carriers to liquid ones: equation 2 outlines how to calculate this dimensionless index figure.

$$1\ \text{Wh kg}^{-1} = 36\ \text{bar L kg}^{-1} = p\,\rho^{-1} = \frac{\text{specific energy [L bar kg}^{-1}\text{] storage medium density } \rho \text{ [kg L}^{-1}\text{]}}{\text{storage pressure } p \text{ [bar]}} \qquad \textbf{(2)}$$

Fifth, that dimensionless number may be understood as a measure for the thermodynamic utility of an energy carrier. This quantity gains meaning if a) expressed as natural logarithm and b) put in relation to Wright's law in the sense of an advantage factor. The relative difference in cost between two energy carrier value chains is then given by the quotient of respective logarithmic advantages. In order to demonstrate that equation 1 is expressed for a generic option (P, x, b) as equation 3 shows.

$$P_x = P_1\, x^{\log_2(b)} \quad \Rightarrow \quad \ln\left(\frac{P_x}{P_1}\right) = \ln(x)\,\log_2(b) \quad \Rightarrow \quad \ln\left(\frac{P_x}{P_1}\right)\frac{\ln(2)}{\ln(x)} = \ln(b) \qquad \textbf{(3)}$$

Equation 3 is fundamental for the per-unit cost development and equation 4 shows an exemplary relative cost comparison between two options (P, x, b) and (Q, x, c) at equal production amount x.

$$\frac{\ln(b)}{\ln(c)} = \frac{\cancel{\ln(2)\,\ln(x)}}{\cancel{\ln(2)\,\ln(x)}}\,\frac{\ln\left(\frac{P_x}{P_1}\right)}{\ln\left(\frac{Q_x}{Q_1}\right)} \qquad \textbf{(4)}$$

Equation 4 shows that Wright's law simplifies in such a comparison: the issue then comes down to determining the values $\ln(b)$ and $\ln(c)$. Intriguingly, no knowledge of any of the actual cost quantities on the right of equation 4 is vital. However, equation 4 needs further refinement as in a value chain only a fraction of the specific energy is made effective: towards that end the efficiency factors β and γ are introduced to b and c as equation 5 shows.

$$\frac{\ln(\beta\, b)}{\ln(\gamma\, c)} = \frac{\ln(\beta) + \ln(b)}{\ln(\gamma) + \ln(c)} \qquad \textbf{(5)}$$

The problem that coefficients β and γ are unknown is mitigated by an stochastic approach of symmetry: Because efficiency factors β and γ are always between 0 and 1, it is sensible to define β and γ each by the value pair of 0.25 and 0.75, yielding a mean of 0.5 with $\Delta\beta_{\max} = \Delta\gamma_{\max} = 0.25$; hence the distances to the centre and the ultimate interval borders are equal. As the quotient of logarithms in equation 5 scales in linear manner, the interval [0.25, 0.75] provides also a measure for the interval [0, 0.5] respective [0.5, 1]. Thus, the quotient of ln(*b*) and ln(*c*) at $\beta = \gamma = 0.5$ is defined the expected value *E*(*X*) as equation 6a shows. Equation 6b displays the regularity *F*(*X*) by which a deviation from *E*(*X*) may occur; its variance is given by the permutations of β and γ (equation 6c).

$$E(X) \coloneqq \frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} \tag{6a}$$

$$F(X) \coloneqq \frac{\ln(\beta\, b)}{\ln(\gamma c)} = \frac{\ln(\beta) + \ln(b)}{\ln(\gamma) + \ln(c)} \qquad | \qquad \beta, \gamma \in \{0.25, 0.75\} \tag{6b}$$

$$\begin{aligned} Var(X) = \sigma^2 = \; & \left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} - \frac{\ln(0.25) + \ln(b)}{\ln(0.25) + \ln(c)}\right)^2 + \left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} - \frac{\ln(0.25) + \ln(b)}{\ln(0.75) + \ln(c)}\right)^2 + \\ & \left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} - \frac{\ln(0.75) + \ln(b)}{\ln(0.25) + \ln(c)}\right)^2 + \left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} - \frac{\ln(0.75) + \ln(b)}{\ln(0.75) + \ln(c)}\right)^2 \end{aligned} \tag{6c}$$

Based on equations 6a and 6c it is possible to relate via the Bienaymé–Chebyshev inequality interval size from *E*(*X*) to the minimum probability at which random values of *F*(*X*) may fall within bounds. While factors β and γ are not exactly random quantities they nonetheless comprise for a value chain of the product of multiple independent process steps efficiencies, hence random is a fair viable approximation. The horizon to any meaningful insight is given by the probability of 50 : 50 which is defined by a $\pm\sqrt{2}\,\sigma$ interval around the expected value; in kind, it is for a probability of 96.0 : 4.0 $\pm 5\,\sigma$ and $\pm 10\,\sigma$ for a 99.0 : 1.0 probability. The 96.0 percent probability solution represents a reasonable trade-off between interval size and probability as abiding the general quality rule of statistics that a confidence interval should in 95 % of all cases enclose the unknown true value. Understanding the two ± intervals around *E*(*X*) as a two-dimensional problem, the final $\mathbb{E}$-value of a distinct probability may be obtained as the quantity of the vector $(E(X) + x\,\sigma)$; $(E(X) - x\,\sigma)$). Equation 7 shows the according base relation.

$$\mathbb{E}_y = \sqrt{\left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} + x\,\sigma\right)^2 + \left(\frac{\ln(0.5) + \ln(b)}{\ln(0.5) + \ln(c)} - x\,\sigma\right)^2} \tag{7}$$

$(x = \sqrt{2}, y = 50)$; $(x = 5, y = 98)$; $(x = 10, y = 99)$

Intriguingly, the issue of unknown total efficiencies of two respective value chains β and γ can be mitigated without any need of detailed knowledge; the comparatively weak boundary prediction capability of the Bienaymé–Chebyshev inequality is in this context an advantage considering the many elastic sub-factors which might contribute to coefficients β and γ.

The results obtained from this approach may be idealized but they are nonetheless of significance because they represent the ideal fundament on which reality-based scenarios may be developed.

3. Results

The thermodynamic utilities *b* of energy carriers in the sense of equation 2 are determined for six cases: diesel fuel, motor gasoline, compressed hydrogen (cH2) at 350 bar and 700 bar, liquefied hydrogen at 1 bar (LH2) and lithium-ion cell (Li-ion). The specific energy of the carbon-fuels is derived on basis of the *tonne of oil equivalent* (*toe*) conversion system as by the International Energy Agency: a *toe* is the energy equivalent of 11.63 MWh. All densities are for the plain energy carriers. The thermodynamic utilities thus obtained are transformed further into their logarithmic advantage ln(*b*): table 1 displays the results.

Table 1 Determination of thermodynamic advantage coefficients for various energy carrier technologies.

	Specific energy		Storage pressure p	Density ρ	Utility b	Advantage ln(b)
	[kWh kg^{-1}]	[L bar kg^{-1}]	[bar]	[kg L^{-1}]		
Diesel fuel[a]	11.24	404640	1	0.840	339897.6	12.7364
Motor gasoline[b]	10.87	391320	1	0.745	291533.4	12.5829
cH2 (350 bar, 288 K)	33.33	1199880	350	0.024	82.3	4.4104
cH2 (700 bar, 288 K)	33.33	1199880	700	0.040	68.6	4.2283
LH2 (1 bar, 20 K)	33.33	1199880	1	0.071	85191.5	11.3527
Li-ion[c]	0.224	8064	1	3.094	24950.0	10.1246

[a, b] 0.96618 toe (diesel), 0.9345 (gasoline) toe https://www150.statcan.gc.ca/n1/pub/57-601-x/00404/4173233-eng.htm

[c] Panasonic NCR18650GA Li-ion as by datasheet: 693 Wh L^{-1}, 224 Wh kg^{-1} ⇒ ρ = 3.094 kg L^{-1}.

The figures of table 1 reveal three noteworthy findings: *First*, the difference between diesel and motor gasoline advantage factors is not very pronounced; the same applies for both compressed hydrogen gas cases. The former result may be seen a trivial reflexion of everyday reality but the latter is in accordance with the findings of a well-to wheel efficiency parameter study from 2005.[5] Consequently, both respective case pairs may be subsumed as C-fuel and cH2. *Second*, despite hydrogen packing a very high nominal specific energy, the actual utility of compressed gas storage emerges last. Within the concept of thermodynamic advantage, a combination of high storage pressure and low medium density can offset the benefits of an apparent high specific energy. This is an instructive, axiomatic assessment and that the principle works also the other way is discernible with the Li-ion example. *Third*, all hydrogen-related results of table 1 are generally too high because of the temperature-dependency of pressure *p*. While the thermal effect of volume expansion may be neglected in the C-fuel and Li-ion cases for the rather small ΔT value and the logarithmic nature of advantage, this is not possible for hydrogen technologies due to the pronounced relation between gas volume, pressure and temperature. A further reason is the vital role of temperature towards modulating/upholding the storage condition of hydrogen. The coefficient of thermal volume expansion α_V is connects via the volume of the density as displayed in equation 8.

$$\text{Utility factor } b = p\,\rho^{-1} = p\,\frac{V}{m} = p\,\frac{V\,(1 + \alpha_V\,\Delta T)}{m} \quad \Rightarrow \quad p' = p\,(1 + \alpha_V\,\Delta T) \quad \Rightarrow \quad b = p'\,\frac{V}{m} \qquad \textbf{(8)}$$

Equation 8 shows that within the concept of thermodynamic advantage a higher temperature spread ΔT vital to energy carrier storage/usage has a reducing effect on its utility *b* which is a matter of immediate clarity. This reduction in utility can be accounted for either by means of an abatement on specific energy or by increasing the nominal storage pressure *p* to *p*' as outlined in equation 8. The hydrogen thermal volume expansion coefficient is assumed as constant and near-ideal with $\alpha_V = (273.07\text{ K})^{-1} = 36.62\cdot10^{-4}\text{ K}^{-1}$. With regard to cH2 storage in type IV tanks,[6] the vessel temperature should remain for safety reasons below 85 °C due to the issue of de-lamination. This marks the upper limit to ΔT. The lower limit is given by fuelling protocol SAE TIR J2601: depending on dispenser type, typical pre-cooling temperatures for 700 bar storage are either –20 °C or –40 °C.[7] These dispenser temperatures apply for 350 bar delivery as well, though pre-cooling temperatures of 0 °C or even above may be in this case viable as well. However, cH2 storage is an average consideration and for that reason the common dispenser temperature of –20 °C is taken as lower limit, yielding a temperature spread ΔT of 105 °C. This equals an increase of 38.44 % or p' values of 969 bar and 485 bar, respectively. In relation to the peak pressures of 875 bar and 438 bar as by SAE J2601 hydrogen fuelling protocol, these figures suggest a 10.7 % or 3.6 kWh (kg H_2)$^{-1}$ penalty on utility for the pre-cooling/compression process which is a reasonable, even somewhat benevolent assessment.[8] Whereas for cH2 hydrogen storage the penalty on utility was accounted for by means of pressure p', it is more straightforward for LH2 storage to put the penalty on specific energy right from the start and reduce it by the about 11.9 kWh (kg H_2)$^{-1}$ the liquefaction process consumes.[9] Furthermore, the boil-off loss respective dormancy issue is to consider, a quite elastic quantity since depending much on externalities such as scale and pressure (e.g. a 350 bar cryo-tank has been used to avoid the issue altogether).[10] The data by Petitpas may serve as base for modelling dormancy,[11] if the initial data are transformed from "percentage lost per kg-amount LH2 transferred" to "residual LH2 watt-day(s)".[8] Polynomial fitting suggests the relative proportion invariant to scaling effects (the zero-order term) to be about 60 %. Thus, in the LH2 case specific energy recedes from 33.33 kWh kg^{-1} to about 0.6 (33.33 – 11.9) kWh kg^{-1} = 12.86 kWh kg^{-1}. Fuel cells of the PEM type operate typically at pressures above 2 bar for (quasi-)stationary purposes while applications of a more dynamic profile such as e.g. electric vehicles (FCEV) require a reservoir pressure between 10 to 15 bar as revealed by the Toyota Mirai FCEV dismantling manual. Typically a safety pressure factor of 2.25 is assumed for hydrogen technologies;[8] so the pressure threshold for LH2 tanks is set between 5 bar and 35 bar. Table 2 shows the above refinements to table 1 values.

Table 2 Adjusted and refined data of table 1.

	Specific energy		Storage Pressure p' [bar]	Density ρ [kg L^{-1}]	Utility b	Advantage ln(b)	
	[kWh kg^{-1}]	[L bar kg^{-1}]					
C-fuel	11.06	398160	1	0.793	315741	12.6627	
cH2 (288 K, 350 bar)	33.33	1199880	485	0.024	59.4	4.0839	3.9933
cH2 (288 K, 700 bar)			969	0.040	49.5	3.9026	
LH2 (20 K)	12.86	462960	5	0.071	6.574	7.909	7.3770
			35		939.1	6.8450	
Li-ion	0.224	8064	1	3.094	24950.0	10.1246	

Table 2 figures allow with equations 6a to 7 the assessment of relative final cost: for cost is assumed to relate reciprocally-linear to advantage, C-fuel is placed in the numerator, table 3 sums the results.

Table 3 Assessment of the relative final cost of C-fuel substitution on basis of table 2 figures and equations 6a to 7.

	C-fuel	cH2	LH2	Li-ion
Advantage ln(b)	12.6627	3.9933	7.3770	10.1246
ln(0.5) + ln(b)	11.9696	3.3002	6.6839	9.4315
$E(X)$	1.0000	3.6269	1.7908	1.2691
Standard deviation σ to $E(X)$	–	1.4716	0.3551	0.1931
$E(X) + \sqrt{2}\,\sigma$ / $E(X) - \sqrt{2}\,\sigma$	–	5.7081 / 1.5457	2.2930 / 1.2886	1.5422 / 0.9960
$\mathbb{E}_{50}$	–	5.9137	2.6303	1.8359
$E(X) + 5\,\sigma$ / $E(X) - 5\,\sigma$	–	10.9849 / –3.7311	3.5663 / 0.0153	2.2346 / 0.3036
$\mathbb{E}_{96}$	–	**11.6013**	**3.5663**	**2.2551**
$E(X) + 10\,\sigma$ / $E(X) - 10\,\sigma$	–	18.3429 / –11.0891	5.3418 / –1.7602	3.2001 / –0.6619
$\mathbb{E}_{99}$	–	21.4343	5.6243	3.2678

4. Discussion

From a first glance, the conception of an thermodynamic utility index might appear a wild move but taking a closer look, it shows that this actually gets all the essential facts of the matter right: *First*, the axiomatic finding that high pressure and/or temperature process steps reduce the practical utility of any energy technology (value chain). *Second*, a rather high medium density in the storage state is beneficial towards the utility of energy technologies whereas managing large volumes is not. *Third*, diesel and motor gasoline are as energy carriers of about equal utility though one might have an advantage over the other with special applications (e.g. tanks). *Fourth*, a well-to wheel efficiency parameter study by Brinkman *et al* from 2005 arrives at the surprising result that it does not make a profound difference whether a 350 bar or 700 bar technology is considered.[5] *Fifth*, regarding cH2 storage, the delivery pressures as by SAE TIR J2601 are substantially higher than the actual storage pressures: 875 bar respective 438 bar. The conception of thermodynamic utility embraces all these points with ease. It may be argued that it is just five coincidences but that cannot be considered a strong argument as these “coincidences” relate in systematic manner to features of fundamental and particular nature alike. The linking to Wright's law is formally admissible because of its empiric character but for the selfsame reason a final compelling proof of truth is not possible; it is likewise true for the statistical approach via the Bienaymé–Chebyshev inequality. However, prerequisites and boundary conditions are clearly marked out. It is certainly debatable whether cost and utility relate reciprocally-linear but it is a viable first-order approximation. Due to the logarithmic-linear relative nature of the approach, any featured error is systematic which prevents things from going entirely out of proportion. It is left to individual judgement to what extent all that may constitute proof.

Regardless, there is evidence for the concept of thermodynamic utility capturing at least part of the regularity to reality: that statement is far from wild guessing. The figures obtained are reasonable enough results although of the three $\mathbb{E}$-values calculated the two outer ones seem to push it a bit towards extremes. The $\mathbb{E}_{96}$ value allegedly balances interval size, probability and insight best.

The results obtained naturally depend on their input parameters and thus are somewhat elastic quantities and e.g. the dormancy penalty on LH2 storage is with 40% rather high (but owed to scope). Yet that is of rather secondary importance: it is the coherence of outcomes in relation to input parameters which makes for the remarkable result: not everything is possible; there are rules and proportions to the obtainable figures, reflecting the principles to reality well. On grounds of the exemplary baselines outlined, further-reaching scenario questions may be transparently evaluated, e.g. what shortage level in C-fuels, extent of subsidy or relative production numbers would be required for cost/price parity. In conclusion, this paper's is not about an exhaustive self-sufficient truth but the provision of a simple, powerful enabling tool whose individual adaption is encouraged.

5. Conclusion

This paper presents evidence for the thesis that the subject of C-fuel substitution is governed by an intrinsic thermodynamic causality beyond economic and human factors, ultimately defining hidden pre-existing ideal baseline thresholds to the achievable in reality.

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7. Conflicts of Interest

There are no conflicts of interest to declare.

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