Orbital-like Standing Waves Using Chladni Plates

Supporting Information - Mathematical Foundations

1 The guitar string – a one-dimensional wave

A uniform string attached at two end points, such as a guitar string of length a, has an amplitude (displacement) $f(x,t)$ that depends on the point in space and on time, and follows the wave equation:

$$\nabla^2 f(x,t) = \frac{1}{v^2}\frac{\partial^2 f(x,t)}{\partial t^2}, \tag{1}$$

where the Laplacian $\nabla = \frac{\partial}{\partial x}$ for the one-dimensional case, and v is the velocity with which the wave propagates along the string. The solution of Equation 1 is subject to the problem's boundary conditions, which in this case are that the function must remain zero at the two end points:

$$f(0,t) = f(a,t) = 0. \tag{2}$$

The two variables x and t act separately on $f(x,t)$ in Equation 1, so we can choose to work with a separable solution $f(x,t) = X(x)T(t)$. The separation of variables can be done in the usual way,[1, 2] resulting in

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{v^2T(t)}\frac{d^2T(t)}{dt^2} = k, \tag{3}$$

where k is a separation constant, written this way for simplicity of the derivation below.

Both the time and the space dependence sides of Equation 3 are second order differential equations in which the second derivative of the function is proportional to the function itself.

Such equations have solutions that are either exponentials or combinations of sine and cosine functions.

For a position trial function that is written as $X(x) = c_1 \sin \beta x + c_2 \cos \beta x$, application of boundary conditions gives:

$$X(0) = c_2 = 0; \tag{4}$$
$$X(a) = c_1 \sin \beta a + c_2 \cos \beta a = 0, \tag{5}$$

leading to $\beta = \frac{n\pi}{a}$ $(n = 1, 2, 3, ...)$, $c_2 = 0$ and $X(x) = c_1 \sin \frac{n\pi x}{a}$, and, in Equation 3, $k = -\beta^2 = -\frac{n^2\pi^2}{a^2}$.

A similar procedure is followed for $T(t)$, but given that there are no boundary conditions for this function, both the sine and the cosine functions survive. Instead, we can use a contracted trigonometric function to describe it: $T(t) = d_1 \cos(\omega t + \phi)$. Plugging $T(t)$ into Equation 3 gives the dependence of the angular frequency ω on the integer n, $\omega = \frac{n\pi v}{a}$.

The separable solution of the wave equation becomes

$$f_n(x, t) = d_1 c_1 \cos(\omega_n t + \phi_n) \sin \frac{n\pi x}{a}. \tag{6}$$

where the subscript n indicates a function or a constant that depends on the integer n. In acoustic terms, $f_1(x, t)$ is the fundamental mode or the first harmonic, with frequency $\nu_1 = \frac{\omega_1}{2\pi} = \frac{v}{2a}$. $f_2(x, t)$ is the second harmonic or first overtone, with frequency $\nu_2 = 2\nu_1$, etc. Two further points can be drawn from Equation 6: one is that each $f_n(x, t)$ has $n - 1$ nodes (points in space where the function is zero), and that the position of the nodes is independent of time (in other words, the functions represent standing waves).

A general solution of the wave equation is then a linear combination of the separable solutions,

$$F(x, t) = \sum_{n=1}^{\infty} f_n(x, t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \phi_n) \sin \frac{n\pi x}{a}, \tag{7}$$

where A_n are expansion coefficients, depending on the order of the harmonic and the initial condition of the function (e.g., how and how hard the guitar string is played). Plucking a guitar string at the middle produces a wave that has a high proportion of the fundamental mode of that string, whereas plucking the string at another location will have an asymmetric combination of frequency modes. The speed of the wave through the guitar string, and thus its frequency, depend on the material of the string and on its tightness.

2 The membrane attached at the sides – A two-dimensional wave

A rectangular membrane follows a wave equation similar to Equation 1, with the modification that the function now depends on two spatial variables: $f(x, y, t)$, and the Laplacian is a two-dimensional operator in this case:

$$\frac{\partial^2 f(x,y,t)}{\partial x^2} + \frac{\partial^2 f(x,y,t)}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 f(x,y,t)}{\partial t^2}. \tag{8}$$

A separable solution is sought $f(x, y, t) = X(x) * Y(y) * T(t)$, and separation of variables is again pursued by inserting this solution into Equation 8. This leads to a set of single-variable equations [1] as follows:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2; \tag{9}$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2; \tag{10}$$

$$\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = -\omega^2, \tag{11}$$

where $k_x^2 + k_y^2 = \omega^2/v^2$. All of the single-variable equations have trigonometric solutions as shown in the one-dimensional case above:

$$T(t) = g_1 \sin(\omega t) + g_2 \cos \omega t; \tag{12}$$

$$X(x) = c_1 \sin(k_x x) + c_2 \cos k_x x; \tag{13}$$

$$Y(y) = d_1 \sin(k_y y) + d_2 \cos k_y y. \tag{14}$$

The boundary conditions $X(0) = X(a) = 0$, $Y(0) = Y(b) = 0$ [equivalent to the full solution boundary conditions $f(0, y, t) = f(a, y, t) = 0$, $f(x, 0, t) = f(x, b, t) = 0$] cancel out the cosine terms in the $X(x)$ and $Y(y)$ solutions, and enforce values for k_x and k_y that are multiples of π/a and π/b, respectively.

The separable solution becomes

$$\begin{aligned} f_{mn}(x,y,t) &= c_1 d_1 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)[g_1 \sin(\omega_{mn}t) + g_2\cos(\omega_{mn}t)] = \\ &= A \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cos(\omega_{mn}t + \phi_{mn}). \end{aligned} \tag{15}$$

These harmonics can be obtained by resonant activation of the attached membrane, and are stationary states. Their nodes are linear, parallel to the membrane sides, at locations where $\sin(x)$ and $\sin(y)$ are zero, with a total of $m+n-2$ nodes for each $f_{mn}(x,y,t)$ harmonic.

The general solution of the wave equation for a rectangular drumhead given by a linear combination of the stationary state functions,

$$F(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(x,y,t). \tag{16}$$

On a **circular drumhead**, vibrational waves are treated in a similar fashion.[3] A polar coordinates Laplacian is used, with separation of variables leading to a radial differential equation (a Bessel equation), an equation for the polar angle, and a time-dependent differential equation. The solution is more complicated than the rectangular case, but there are again two boundary conditions (that the radial function should be zero at the edge of the membrane and that the polar function should have circular symmetry) leading to the emergence of two quantum numbers. On the circular membrane, nodes can be either circular (radial nodes, almost equally spaced) or linear along diameters of the membrane.

The wave equation for a **vibrating plate with free edges**, such as that used in our Chladni apparatus, is identical to that presented for the stretched membranes above.[4] However, boundary conditions are nonzero at the free edges of the plates, as these edges are free to vibrate. For a rectangular (a,b) plate, the boundary conditions become complicated:[4]

$$\frac{\partial^2 f(x,y,t)}{\partial x^2} + \nu_p \frac{\partial^2 f(x,y,t)}{\partial y^2} = \frac{\partial^3 f(x,y,t)}{\partial x^2} + (2-\nu_p)\frac{\partial^3 f(x,y,t)}{\partial x \partial y^2} = 0 \text{ when } x = \pm a; \tag{17}$$

$$\frac{\partial^2 f(x,y,t)}{\partial y^2} + \nu_p \frac{\partial^2 f(x,y,t)}{\partial x^2} = \frac{\partial^3 f(x,y,t)}{\partial y^2} + (2-\nu_p)\frac{\partial^3 f(x,y,t)}{\partial x^2 \partial y} = 0 \text{ when } y = \pm b, \tag{18}$$

where ν_p is the plate material's Poisson ratio.

The solution of the elementary differential equations is more complex than in the case of the rectangular plate with free edges, because of the nontrivial boundary conditions, and can be found elsewhere.[4] For the purpose of the current application, it is sufficient to note that the physical characteristics of these harmonics are very similar to those of the circular membrane.

3 The electron in a Hydrogen atom – a three-dimensional wave

The solution of the wave equation for the Hydrogen atom is available in any Physical Chemistry or Quantum Chemistry textbook. Broad lines are presented here for comparison to the classical wave solutions presented above.

The Hydrogen atom consists of a proton orbited of an electron. The mass difference between the two particles leads to the Born-Oppenheimer (BO) approximation: As a consequence, the system is described as being composed of the proton as a stationary heavy particle at the origin, and the electron as a moving particle of reduced mass μ. The two particles interact through a Coulomb potential $V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$, where r is the position of the electron in the proton-centered coordinate system.

The time-dependent Schrödinger equation is the wave equation describing the behaviour of the system:

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi(r,\theta,\phi,t) + V(r)\Psi(r,\theta,\phi,t) = -i\hbar\frac{\partial}{\partial t}.\Psi(r,\theta,\phi,t), \tag{19}$$

where spherical coordinates are used to reflect the symmetry of the problem. Since the Hydrogen atom potential is independent of time in the BO approximation, a separable solution $\Psi(r,\theta,\phi,t) = \psi(r,\theta,\phi)T(t)$ can be sought, leading to two separate parts of the Schrödinger equation:

$$-i\hbar\frac{\partial}{\partial t}T(t) = E\,T(t); \tag{20}$$

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi), \tag{21}$$

where the separation constant E is the energy of the system, and the time-dependent part of the separable solution is $T(t) = e^{-iEt/\hbar}$.

The time-independent part of the wavefunction is also separable, $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$. Note that so far, the problem is similar to that of the acoustic waves on circular plates, with two distinctions besides the dimensionality of the problem: (i) We chose the exponential form of the solution of Equation 20 for mathematical convenience, and (ii) There is a potential acting on the wave, $V(r)$.

There are likewise similarities and differences between the solutions of the wave equations in the two systems. The radial wavefunction in the quantum problem is the solution of a ordinary differential equation that includes the potential term, so that the radial functions are a series of associated Laguerre functions, instead of the Bessel function solution in the classical case. Both have oscillatory behaviour, but amplitudes and the spacings between nodes are different. The angular functions for both systems are eigenfunctions of the angular part of the squared Laplacian (although we are comparing a 2D problem to a 3D problem in this case). This is the main reason behind the similarity of the nodal shapes reported in the manuscript.

As mentioned above, the Laplacian in spherical coordinates has a relatively complicated expression, and it is not the point of this work to detail the textbook solution of the separable equation. The boundary conditions are given by the requirements that the wavefunction should be finite, well behaved, and single valued at all points. This places restrictions on the radial function at its boundaries $[rR(0) = rR(\infty) = 0]$, on the angular function at the $[0,2\pi]$ boundaries of the azimuthal angle $[Y(\theta,\phi) = Y(\theta,\phi+2\pi)]$, and forces the selection of Legendre polynomials as solutions of the θ-dependent ordinary differential equation. These boundary conditions produce quantization of the energy, the orbital angular momentum and its z projection. Because the boundary conditions force oscillatory behaviour to fit within the boundaries, the associated quantum numbers are related to the number and geometry of nodes in the resulting wavefunction.

The total number of nodes is determined by the principal quantum number, whereas the nature of the nodes is determined by the orbital quantum number. s orbitals have only radial nodes, whereas orbitals with $l > 0$ also have angular nodes. Radial nodes are spherical, while angular nodes can be planes or cones. These nodes are normally represented in 2 D projection (see the Orbitron webpage for example, http://winter.group.shef.ac.uk/orbitron/AOs/3p/wave-fn.html), in which case they take shapes analogous the Chladni patterns discussed above.

References

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[3] D. Yong. The Wave Equation and Separation of Variables. Technical report, IAS. Park City Mathematics Institute, 2003.

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