How accurate are static polarizability predictions from density functional theory? An assessment over 132 species at equilibrium geometry.†

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Static polarizabilities are the first response of the electron density to electric fields, and are therefore important for predicting intermolecular and molecule-field interactions. They also offer a global measure of the accuracy of the treatment of excited states by density functionals in a formally exact manner. We have developed a database of benchmark static polarizabilities for 132 small species at equilibrium geometry, using coupled cluster theory through triple excitations (extrapolated to the complete basis set limit), for the purpose of developing and assessing density functionals. The performance of 56 popular and recent functionals are also assessed, which indicates that double hybrid functionals perform the best, having RMS errors in the range of 2.5-3.8%. Many hybrid functionals also give quite reasonable estimates with 4-5% RMSE. A few meta-GGAs like mBEEF and MVS yield performance comparable to hybrids, indicating potential for improved excited state predictions relative to typical local functionals. Some recent functionals however are found to be prone to catastrophic failure (possibly as a consequence of overparameterization), indicating a need for caution in applying these.

1 Introduction

The vast majority of present day electronic structure calculations employ some flavor of Kohn-Sham (KS) density functional theory (DFT)1–4. The exact functional mapping ground state electron density to ground state energy however remains computationally inaccessible, despite proof of its existence5. This has led to development of hundreds of density functional approximations (DFAs) of varying complexities through strategies ranging from a sole focus on satisfaction of exact physical constraints6–8 to exclusively fitting to benchmark data9,10. Empirically fitted functionals are expected to be extremely accurate within the datasets employed for fitting, but they can be overparametrized to a point where performance outside of that comfort zone becomes rather suspect11–14. Conversely, functionals with none or few fitted parameters may not be as accurate for specific datasets but are believed to be much more transferable to systems not employed in the development process. Many modern functionals have consequently been developed through approaches that draw from both extremes15,16, in the hope of having an acceptable balance between fit accuracy and transferability.

In general, a DFA should be transferable to any system of interacting electrons subjected to some external one particle potential ν(r). Isolated molecules often employed in fitting functionals satisfy the above condition (with ν(r) being the electrostatic potential of the nuclei) but they represent only a small fraction of systems where DFT is applicable. Molecular systems studied with DFT in fact often interact with external fields in chemically relevant scenarios—such as light-matter interactions or in condensed phase. The behavior of density functionals outside the traditional zero external field conditions thus has direct relevance to simulations of spectroscopy, dielectric solvation models17 and QM/MM, amongst other things.

The simplest possible external field is a constant electric field ⃗E, and the corresponding responses of the energy E(⃗E) are well known molecular properties18. The first response of the zero-field energy E(0) to ⃗E is the dipole moment ⃗μ = − ( ∂E ⃗E ∂⃗E ) ⃗E=0 , which is a simple global measure of the accuracy of electron density in a polar molecule. We studied the accuracy of DFT in predicting ⃗μ in an earlier work for the purpose of gauging the quality of DFT densities13, and will not consider it further here.

The dipole however contains no information about the response of the ground state wave function to the electric field as it can be computed from the zero-field density alone. The first property to contain information about the wave function response is the static polarizability tensor α which is the second response of the zero field energy to the external field. The components αij are given by − ( ∂2E ⃗E ∂⃗Ei ∂⃗Ej ) ⃗E=0 , which allows us to express it in terms of excited electronic states at zero external field using perturbation theory. Specifically, given a ground state |Ψ0⟩ with energy E0 and excited states { |Ψn⟩ } with energies { En } at ⃗E = 0, we have:

Equation (1)

where ⃗μ is the dipole operator. The ability of a functional to predict static polarizabilities therefore not only reveals how the electron density responds to an external field, but also gives indirect information about the treatment of dipole allowed excited states by the functional. This is a coarser measure than linear response TDDFT19 in that the information about a single excited state cannot be disentangled, but it is formally exact as it does not involve use of the adiabatic local density approximation (ALDA)20. Polarizability predictions therefore indirectly measure both the transferability of a functional and the quality of excited state predictions (within the limits of the Hohenberg-Kohn theorem), making it useful for assessing functional quality. Fitting functionals to polarizabilities should also assist in ensuring correct response of E to external electric fields.

There however does not appear to be a large database of benchmark static polarizabilities which could be used for either fit-
tering density functionals to, or for assessing their performance. Past assessments have often utilized experimental values, which may contain nuclear quantum effects that no accurate electronic structure method should reproduce within the Born-Oppenheimer approximation. Some studies also appear to suggest that many experimental polarizabilities may contain substantial errors, making a comparison to benchmark wave function theories necessary for a truly fair assessment of the accuracy of polarizability predictions from DFAs. There have been several such studies, but they appear to either consider only a narrow range of molecules or use an apparently inadequate level of theory for reference values. The closest approximation to a benchmark database of polarizabilities appears to be CCSD(T)/aug-cc-pVTZ values for 145 organic molecules compiled by Wu, Kalugina and Thakkar for assessing the performance of 34 DFAs. We however feel that these values may not be of benchmark quality on account of missing the complete effect of core excitations and basis set incompleteness errors, which can be substantial for dipole moments and polarizabilities (as will be shown later). Ref also only considered closed shell molecules and did not assess the performance of double hybrid functionals, which are known to be excellent for predicting energetics and densities.

We have consequently developed a database of the diagonal components of static polarizabilities of 132 species at equilibrium geometry, using coupled cluster singles and doubles with perturbative triples (CCSD(T)) extrapolated to the complete basis set limit (CBS). 75 of the species have a stable spin-unpolarized Hartree-Fock (HF) solution while the remaining 57 have a stable spin-polarized HF solution either due to open-shell character or spin-symmetry breaking. The not spin-polarized (NSP) subset is expected to be relatively well described by conventional electronic structure methods, but the spin-polarized (SP) subset has the potential to be more challenging. This database is used to assess the performance of 56 popular and recent density functionals, along with HF, second order Møller-Plesset perturbation theory (MP2) and coupled cluster singles and doubles (CCSD). This dataset should also be useful for future functional development and assessment. In particular, it permits evaluation of the accuracy with which various DFAs predict individual diagonal elements of the static polarizability tensor, instead of just their average (where some cancellation of errors may occur) as is the case in Ref. The high computational cost of large basis set coupled cluster calculations however constrained us to only consider species smaller than most studied in Ref, though we feel general features from this dataset ought to be transferable to larger systems where KS-DFT can be applied. It is however quite likely that somewhat different behavior is observed for systems with substantial multi-reference character, like long chains where the accuracy of KS-DFT polarizability calculations have long been suspect.

2 Computational Methods

All the calculations were done using a development version of Q-Chem 5. Most of the equilibrium geometries employed in the study were either obtained from experimental values in the NIST Computational Chemistry Database or were optimized with MP2/cc-pVTZ if experimental geometries were unavailable. The exceptions to this general principle are BH$_2$F and BH$_2$Cl, whose experimental geometries were taken from Refs and respectively, as well as the non-covalent complexes FH-OH and H$_2$O-Li, whose geometries were obtained from the TAD database. The all the geometries employed in the present study are provided in the Supporting Information, along with the source.

Polarizabilities were obtained from finite differences using a central three point formula $\alpha_i = \frac{E(\varepsilon_i) + E(-\varepsilon_i) - 2E(0)}{\varepsilon^2}$, which should be correct to $O(\varepsilon^2)$. The field strength $\varepsilon$ was set to 0.01 a.u. for most species, but a few s block atom containing species with large polarizabilities appeared to have large higher-order responses (made evident by a $>0.5\%$ shift in HF polarizability estimates on changing $\varepsilon$ from 0.01 to 0.005 a.u.) that necessitated use of $\varepsilon = 0.001$ a.u. for that subset. This smaller field however was not universally employed due to larger risk of contamination from numerical error, especially in species without a highly polarizable s block atom where the HF polarizability estimate was shifting very little on halving the field strength to 0.005 a.u. any-where ($<0.2\%$ for the vast majority). A full list of field strengths used for each molecule is given in the Supporting Information.

HF polarizabilities were obtained from spin unrestricted calculations with the aug-cc-pCVQZ basis, which appeared to be close to the CBS limit (0.2% RMS deviation versus the equivalent aug-cc-pCVQZ numbers). Stability analysis was performed at the aug-cc-pCVQZ level to ensure all solutions were at a minima. Spin unrestricted DFT calculations were done with the aug-pc-4 basis for functionals spanning Rungs 1-4 in Jacob’s Ladder and it was assumed that such 5$\zeta$ basis results ought to be essentially at CBS as well (though behavior for dipole moments suggest that this may not strictly be true for all functionals). Local xc integrals were calculated over a radial grid with 99 points and an angular Lebedev grid with 590 points for all atoms, while non-local VV10 correlation was calculated over an SG-1 grid (which consists of a subset of points employed in a grid with 50 radial and 194 angular points). Stability analysis was done at the aug-pc-2 level to determine which SCF solutions were potentially unstable, and the problematic aug-pc-4 cases were reoptimized to ensure that the energy was at a minimum.

No orbitals were held frozen for any correlated wave function theory (WFT) calculation. All coupled cluster calculations (CCSD/CCSD(T)) were done employing unrestricted orbitals, but known N-representability violations in unrestricted MP2 led us to calculate both unrestricted (henceforth referred to as MP2) and restricted (RMP2) values for comparison. The CBS limit was obtained by extrapolating the correlation component of $\alpha_i$ from aug-cc-pCVSZ and aug-cc-pCVQZ results for the smaller species; and from the aug-cc-pCVQZ and aug-cc-pCVTZ results for the remainder of the dataset. The extrapolation was done via:

$$\left(\alpha_i\right)_{\text{corr}} = \left(\alpha_i\right)_{\text{pot}} + A/n^3$$

which is known to be accurate for dipole moments. Eqn 2 was found to be accurate to $\approx 0.1\%$ in predicting aug-cc-pCVSZ
polarizabilities of some of the smaller species from aug-cc-pCVQZ and aug-cc-pCVTZ results, indicating that it was reasonably accurate for polarizabilities as well.

The superior performance of RMP2 over unrestricted MP2 led us to only perform calculations with restricted orbitals for double hybrid functionals, with no orbitals held frozen here as well. The xc integrals for double hybrid functionals were calculated using the same grids as all other functionals. The non-MP2 component of \( \alpha \) for double hybrid functionals was assumed to be essentially at the CBS limit when the aug-cc-pCVSZ basis was employed (0.2 – 0.3% RMS deviation versus the equivalent aug-cc-pCVQZ numbers). The MP2 correlation component was extrapolated to CBS from aug-cc-pCVQZ and aug-cc-pCVTZ results, using Eqn 2. Calculations employing the XYZJ-OS \(^{54}\) functional were accelerated using the RI approximation\(^{55}\) with the riMP2-cc-pVTZ auxiliary basis\(^{56,57}\) for aug-cc-pCVTZ calculations and riMP2-cc-pVQZ auxiliary basis\(^{57}\) for aug-cc-pCVQZ calculations.

The error in \( \alpha_i \) against the reference value \( \alpha_{i,ref} \) was defined to be \( \frac{\alpha_i - \alpha_{i,ref}}{\alpha_{i,ref}} \times 100\% \), which is a purely relative error. This was used to calculate root-mean squared, mean and maximum absolute errors for each of three diagonal components of the \( \alpha \) tensor, which are listed in the Supporting Information. The components themselves have very similar relative error values, and so we only report the root mean square error (RMSE), mean error (ME) and maximum absolute error (MAX) over all the species and over the three diagonal components, in the manuscript. In other words, if \( \epsilon_{i,n} \) is the error in \( \alpha_i \) for species \( n \), we have:

\[
\text{RMSE} = \sqrt{\frac{1}{3N} \sum_{i=x,y,z} \sum_{n=1}^{N} \epsilon_{i,n}^2} \tag{3}
\]

\[
\text{ME} = \frac{1}{3N} \sum_{i=x,y,z} \sum_{n=1}^{N} \epsilon_{i,n} \tag{4}
\]

\[
\text{MAX} = \max \left( \{ \epsilon_{i,n} \mid i \in \{x,y,z\} \text{ and } 1 \leq n \leq N \} \right) \tag{5}
\]

where \( N \) is the total number of species. The difference between the RMS errors for the \( \alpha_i \)'s with the largest and smallest RMS errors (DIFF) is also reported, in order to determine if performance varies considerably along different axes.

### 3 Results

The error metrics defined in the preceding section were used to evaluate the performance of HF, MP2, CCSD and 56 DFAs against the CCSD(T) benchmark. The errors for all the electronic structure methods are given in Table 1, while Table 2 contains the benchmark average static polarizabilities (\( \alpha = \alpha_{xx} + \alpha_{yy} + \alpha_{zz} \)) for all 132 species, along with the values predicted by the functional with the lowest cumulative RMSE in each rung of Jacob's ladder. The full list of all errors for all methods is provided in the Supporting Information, along with all computed \( \alpha_i \) polarizability components.

#### 3.1 Performance of Wave function Theory

CCSD proves to be the most effective method for predicting dataset polarizabilities overall, producing an RMSE of 1.62% and giving very similar performance over the NSP and SP subsets. This is partly on account of a conscious choice to only include species with maximum deviation less than 7% between CCSD and CCSD(T) \( \alpha \) in the dataset, in the hope that this would restrict the dataset to species where higher order excitations like exact triples or quadruples would not be necessary to obtain benchmark quality numbers. A large portion of the CCSD error appears to stem from systematic underestimation (ME is \(-1.2\%\) due to the absence of correlation energy from connected triples. It is therefore possible that CCSD (and potentially even CCSD(T)) may not be adequately accurate in predicting benchmark quality polarizabilities outside of our carefully curated dataset, necessitating incorporation of higher order excitations into the coupled cluster scheme. The closeness between CCSD and CCSD(T) values for our dataset, however, indicate that this is not likely to be a concern in the present case.

MP2 performs very well over the NSP species, yielding a small RMSE of 2.17% that compares very favorably to the equivalent CCSD RMSE of 1.55%. This behavior is consistent with the excellent performance of MP2 in predicting polarizabilities of closed shell organic molecules noted in Ref [29]. N-representability violations\(^{52}\) in unrestricted MP2 however leads to catastrophically poor performance (including a few negative \( \alpha_i \)'s!) over the SP dataset, resulting in a rather high RMSE of 11.12% over the entire dataset. Switching over the restricted MP2 (RMP2) only partially ameliorates the problem by bringing the RMSE down to 8.14%, which is still quite large relative to many density functionals. The large SP subset RMSE likely stems from RMP2 operating on a restricted HF reference that is higher in energy than the spin-polarized solution, which leads to extremely large errors for some species like O\(_2\) (predicted \( \alpha_{zz} = 6.732 \text{ Å}^3 \) vs benchmark value of 2.263 \( \text{Å}^3 \))—despite lack of N-representability violations. MP2 therefore is likely to be a poor choice in general for predicting polarizabilities of spin-polarized species, although it remains an excellent choice for spin unpolarized species.

HF gives quite poor performance, yielding an RMSE of 8.45%. Lack of correlation energy in HF theory typically leads to larger excitation energies (as can be seen in CIS\(^{20}\)), which leads to larger denominators in Eqn 1 and consequently smaller \( \alpha_i \). This systematic underestimation however is likely only part of the problem, as the ME is a relatively small -3.04% vs the RMSE, and the extra error likely stems from the poor quality of the method itself. Interestingly enough, restricted HF (RHF) has a slightly smaller RMSE of 7.6% while the ME remains essentially unchanged, suggesting that the constraint of spin restriction eliminated some minor sources of non-systematic error. It is also interesting to note that the RHF RMSE is slightly lower than the RMP2 RMSE, indicating that the expensive perturbation theory correction did little to help for SP species.
### 3.2 Performance of DFT

Table 1 reveals a number of general features that are likely to be transferable beyond our specific dataset. It is firstly striking that all but four functionals have a positive ME. This systematic overestimation of polarizabilities for most functionals can be rationalized in terms of self-interaction error resulting in softer, more polarizable densities. The positive ME can also be interpreted in terms of spin contamination, leading to a smaller denominator in Eqn 1: most DFAs tend to underestimate excitation energies (with both TDDFT and to a lesser extent, SCF type approaches), leading to a smaller denominator in Eqn 1 and consequently larger polarizabilities. The values in Table 1 are consistent with this interpretation, with two of the three functionals producing negative ME containing 50% or more exact exchange. MVSh is the exception as it contains only 25% exact exchange, but this is likely on the account of the base MVS functional already having a very low ME of 0.57% despite being a local functional.

It is also important to note that the best DFA in each rung of Jacob's ladder has a lower RMSE than the best DFAs in the rungs below, which is likely a consequence of the additional physical content at each rung. However, not all functionals employ that extra physical content optimally, as can be seen from the best Rung 3 functionals like mBEEF and MVS outperforming widely used hybrid functionals like B3LYP and M06. There is in fact substantial overlap in accuracy between Rungs 3 and 4, indicating that exact exchange is not always necessary to improve polarizability predictions.

### Table 1 RMS errors (RMSE), Mean errors (ME) and Maximum absolute errors (MAX) for electronic structure methods over the dataset, expressed as percentages. The RMSE of the spin-polarized (SP) and non-spin polarized (NSP) subsets of the dataset are also reported separately. Additionally, the difference between the RMS errors of the components α, β and the largest and smallest RMS errors is reported under DIFF. WFT stands for wave function theory, LSDA for local spin density approximation, GGA for generalized gradient approximation, mGGA for meta-GGA, hGGA for hybrid GGA, hmGGA for hybrid meta-GGA, dhGGA for double hybrid GGA and dhmGGA for double hybrid meta-GGA.

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Table 1 RMS errors (RMSE), Mean errors (ME) and Maximum absolute errors (MAX) for electronic structure methods over the dataset, expressed as percentages. The RMSE of the spin-polarized (SP) and non-spin polarized (NSP) subsets of the dataset are also reported separately. Additionally, the difference between the RMS errors of the components α, β and the largest and smallest RMS errors is reported under DIFF. WFT stands for wave function theory, LSDA for local spin density approximation, GGA for generalized gradient approximation, mGGA for meta-GGA, hGGA for hybrid GGA, hmGGA for hybrid meta-GGA, dhGGA for double hybrid GGA and dhmGGA for double hybrid meta-GGA.
to, or lower than the SP RMSE for most hybrid functionals (despite the latter using unrestricted orbitals) and mark a substantial improvement over RMP2. It is therefore likely quite safe to use double hybrid functionals with restricted orbitals for spin-polarized systems at equilibrium geometry, unlike RMP2 (which gives quite poor performance for some species like $\text{O}_2$).

$\text{PBE50}^{106}$ proves to be the best Rung 4 functional with 3.71% RMSE—largely as a consequence of the functional employing a large fraction of HF exchange (50%), which eliminates the systematic overestimation of polarizabilities present in $\text{PBE7}$ and $\text{PBE0}^{95}$ and indeed, results in a small bias towards underestimating polarizabilities (ME of -1.1%). $\text{SCAN0}^{67}$ is the next best hybrid functional with 3.98% RMSE and it similarly improves upon the already low RMSE of the base SCAN functional (5.31%)

**Table 2** Benchmark CCSD(T)/CBS average static polarizabilities (in $\text{Å}^3$) for all 132 species at equilibrium geometry, along with the values predicted by the functional with lowest cumulative RMSE at each rung of Jacob's ladder. These functionals are SPW92 (Rung 1), N12 (Rung 2), mBEEF (Rung 3), PBE50 (Rung 4) and XYZG-JS (Rung 5) respectively. We counted the H atom as a NSp species despite the odd electron count as the alpha and beta orbitals are formally the same.
by reducing the ME through the 25% exact exchange component. Other hybrid functionals like \( \omega B97X-V \), \( \omega M05-D \), PW6B95 and PBE0 also yield quite good performance with around 4.5% or so RMSE, and there are several others that have errors in the 5-7% range. Hybrid functionals overall also have somewhat larger RMSE over the SP subset than the NSP one, showing that the former are more challenging. The performance gap however is often quite small (<1%), indicating that this is not a major issue for many functionals. BMR\(^{82}\) however proves to be a major exception, with the SP species increasing the total RMSE to 9.49% relative to the excellent NSP subset RMSE of 3%. SCAN0, B97-2\(^{59}\), SOGGA11-X\(^{100}\) and MN15\(^{74}\) are also impacted by the NSP/SP performance difference, though to a much lesser extent. It is also interesting that hybrid meta-GGAs (hmGGAs) do not appear to outperform hybrid GGAs (hGGAs), despite meta-GGAs (mGGAs) being much more accurate than GGAs. This suggests that the vastness of the hmGGA functional space has not been optimally utilized for improving polarizability predictions, which is similar to behavior observed for dipole moments\(^{13}\).

It is therefore noteworthy that many mGGAs (mBEEF, MVS and SCAN in particular) are essentially as good as hybrid functionals at predicting polarizabilities, which is quite unlike the behavior observed for dipole moments\(^{13}\). These three functionals also do not have particularly large systematic biases relative to hybrid functionals (as reflected by their ME), indicating that this particular manifestation of self-interaction error might be rather minimized. It would therefore be interesting to examine whether these functionals are comparable to hybrids for excitation energy predictions. The case of MVS is rather interesting in particular as its hybrid variant MVSh predicts worse polarizabilities (unlike other similar local/hybrid pairs like SCAN/SCAN0 or TPSS\(^{8}/\)TPSSH\(^{77}\)) due to MVS already having a very low systematic bias.

Overall, there is no great chasm in performance between local functionals and hybrids (unlike in dipole moments), with several less accurate mGGAs like MS2\(^{72}\) and M06-L\(^{78}\) giving performance comparable to many hybrid functionals, and some others like TPSS and B97M-V\(^{83}\) being not much worse. The only potentially problematic aspect is the larger gap between NSP and SP RMSEs relative to hybrid functionals, but the actual errors remain fairly low. There is however a rather wide gulf separating mGGAs from GGAs, with the latter being rather poor at predicting polarizabilities relative to mGGAs. Aside from the special case of SOGGA11\(^{73}\) however, GGAs represent an improvement over LSDA but nonetheless ought not be employed to estimate polarizabilities as mGGAs make significantly better predictions for the same computational complexity.

Coming to the matter of poor performers, we find that the worst double hybrid functional B2PLYP\(^{96}\) actually does not have particularly bad RMSE (4.36%) relative to Rungs 3 and 4. Its performance is quite poor relative to its Rung 5 brethren, but it nonetheless represents a significant improvement over related functionals like B3LYP and BLYP in rungs below. The worst hybrid functionals give much less satisfactory performance with RMSE \( \approx 10\% \) but are still better than LSDA. The worst GGA (SOGGA11) and mGGA (M11-L\(^{89}\)) however are worse than Rung 1 LSDA, suggesting overparametrization has led to a subsequent decrease in accuracy. SOGGA11’s performance in particular is exceptionally poor for even simple species like \( \text{H}_2 \) and HCl, to say nothing of more challenging cases like Li\(_2\). This functional also has a rather poor basis set convergence for dipole moments\(^{13}\), suggesting that the aug-cc-p-4 numbers may not be adequately close to the CBS limit. SOGGA11 therefore should not be used for molecular property predictions. Paradoxically however, the related SOGGA11-X hybrid functional performs quite well, suggesting that there may exist some scope for improving SOGGA type functionals.

It is also interesting to note that the best performing functionals have small DIFF values (\( \approx 1\% \) or lower). ME for individual \( \alpha_i \) are also quite similar in value, suggesting that these functionals are about equally as effective in estimating \( \alpha_{xx,yy,zz} \). This is encouraging, as it suggests that good functionals would predict accurate static polarizabilities on account of getting individual components correct, instead of relying on cancellation of errors between them.

Several of the poor performing functionals however have quite large DIFF values, largely as a consequence of catastrophic failure in predicting \( \alpha_{zz} \) (polarizability along the bond axis) for certain s-block diatomics like Li\(_2\) or NaLi, while getting more reasonable estimates for the components perpendicular to the bond axis.

<table>
<thead>
<tr>
<th>Species</th>
<th>Subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Be</td>
<td>SP</td>
</tr>
<tr>
<td>H</td>
<td>NSP</td>
</tr>
<tr>
<td>CN</td>
<td>SP</td>
</tr>
<tr>
<td>C(_2)H</td>
<td>SP</td>
</tr>
<tr>
<td>H(_2)O-Li</td>
<td>SP</td>
</tr>
<tr>
<td>Na</td>
<td>SP</td>
</tr>
<tr>
<td>Li(_2)</td>
<td>SP</td>
</tr>
<tr>
<td>NaLi</td>
<td>SP</td>
</tr>
</tbody>
</table>

Table 3 The 8 most difficult species in the dataset. These were selected on the basis of the first quartile of RMS DFA errors for each species (the RMS being taken over the three components). The selected species giving \( \geq 5\% \) first quartile error, which corresponded to a break in the distribution. Coincidentally, this number is also comparable to typical Rung 4 RMSEs.

3.3 Challenging Species

The eight most challenging species for DFT polarizibility predictions are listed in Table 3, roughly in descending order of difficulty. Most are NSP species, which is entirely unsurprising on account of the best performing functionals being double hybrids that are constrained to act on restricted references alone. The two most challenging species in the dataset by far are the H and Be atoms, which is somewhat surprising on account of their simplicity. The polarizabilities for both are systematically overestimated by \( \approx 10\% \) or more, although the errors for H decrease on increasing the fraction of exact exchange in functionals (expectedly, as HF is exact for the H atom), suggesting that self-interaction error is to blame in this case. On the other hand, Be is likely challenging due to multi-reference character (HF breaks spin-symmetry despite Be formally being a closed-shell singlet atom) and not self-interaction, as the errors remain similar over functionals with different exact exchange fractions. The behavior for both atoms
is quite troubling as it indicates that thirty years of functional development involving many comparisons and fits to such simple atoms nonetheless fails to predict response under a constant electric field. We recommend that the zero external field atomic energies typically used for fitting functionals be augmented with energies and densities for atoms subjected to diverse field strengths, in order to generate a more complete description of such so-called ‘perfect norms’.

Other challenging species are the triple bonded C₂H and CN radicals, and highly polarizable alkali atom containing species like NaLi. Their presence on the list is unsurprising, especially since the latter poses a major challenge to functionals with regards to density predictions. MN12-L and MN12-SX in particular fail catastrophically in predicting properties for such alkali atom containing species, indicating a need for caution in applying these Minnesota functionals to s block elements.

It is also interesting to note that the most challenging species in Table 3 are not amongst the largest species in the dataset. KS-DFT is known to have significant difficulties in predicting α for long chains, although none of the molecules studied were anywhere close to the problematic length scales. The possibility of different behavior on very large length scales therefore cannot be ruled out, especially if the systems have substantial multiple-reference character like long polyene or acene chains.

<table>
<thead>
<tr>
<th>Method</th>
<th>RMSE CVQZ/CTVZ</th>
<th>ME CVQZ/CTVZ</th>
<th>MAX CVQZ/CTVZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCSD(T)</td>
<td>0.46/1.33</td>
<td>0.16/-0.12</td>
<td>2.54/9.14</td>
</tr>
<tr>
<td>CCSD</td>
<td>0.44/1.19</td>
<td>0.21/-0.02</td>
<td>2.08/8.05</td>
</tr>
<tr>
<td>RMP2</td>
<td>0.43/1.42</td>
<td>0.05/-0.37</td>
<td>2.76/9.55</td>
</tr>
<tr>
<td>HF</td>
<td>0.19/1.04</td>
<td>-0.03/-0.6</td>
<td>1.43/6.86</td>
</tr>
</tbody>
</table>

Table 4 Percentage differences between the CBS estimate and aug-cc-pCV(Q/T)Z α, for WFT methods. RHF has essentially same basis set convergence as HF, while MP2 basis-set convergence rate is unreasonably slow due to N-representability violations. A full table with all deviations is supplied in the Supporting Information.

3.4 Basis Set Convergence

HF polarizabilities converge somewhat slowly (relative to behavior seen for dipole moments) on increasing basis set size. The RMS deviation between aug-cc-pCV5Z and aug-cc-pCVQZ numbers is 0.2%, while the equivalent deviation between aug-cc-pCV5Z and aug-cc-pCVTZ is 1%. This indicates that the aug-cc-pCV5Z are likely essentially at the CBS limit, but the aug-cc-pCVTZ numbers are not an optimal replacement. No detailed investigation of basis set convergence of DFT polarizabilities were carried out, but it is not unreasonable to expect behavior similar to HF for all but a few ill behaved functionals that are notorious for slow basis set convergence. This would indicate that quadruple zeta basis numbers might be sufficient close to CBS for practical purposes, but triple zeta basis numbers are likely insufficiently converged.

Correlated WFT estimates for polarizability converge more slowly with increasing basis set size, but does so in a relatively well described manner given by Eqn 2. A comparison between the CBS estimate and aug-cc-pCVTZ/aug-cc-pCVQZ numbers (as given in Table 4) reveals that neither finite basis is an acceptable estimate, and extrapolation almost essential to have benchmark quality numbers. This further indicates that the triple zeta CCSD(T) estimates in Ref [29] would benefit from some refinement (which ought to also incorporate a more complete description of excitations out of the core).

4 Conclusion

In summary, we find that it is possible to get quite accurate static polarizability estimates for small and medium sized species from several DFAs, although most seem to systematically overestimate the values on account of self-interaction error. The best performing functionals are double hybrids, with RMSE in the range of 2.5-3.8%—placing them within striking distance of CCSD (which has 1.61% RMSE). A few (XYGJ-OS and XYG3 in particular) are essentially as good as CCSD for non spin-polarized (NP) species, but their overall performance is worsened on account of being constrained to a restricted reference for even spin-polarized (SP) systems, in order to avoid N-representability breakdowns. Double hybrid functionals employing an orbital optimized MP2 like formalism therefore could lead to significant further improvement in prediction of polarizability (and other molecular properties), as such N-representability violations would not be a concern.

Hybrid functionals also give quite decent performance, with several yielding RMSEs between 4-5%. Truly exceptional performance however is obtained from the mbEEF and MVS mGGAs, which give hybrid functional level performance at a much lower computational cost. This suggests that these functionals are able to minimize some aspect of self-interaction error, and indicate they may yield better excited state predictions than what might typically be expected of local functionals. Other mGGAs like SCAN and M06-L also give satisfactory performance, indicating that there is no large difference in performance between Rungs 3 and 4 of Jacob’s ladder. It would be interesting to see if similar features transfer over to long carbon chains where large fractions of exact exchange have long been held to be essential for decent polarizability predictions but that is beyond the scope of our present study due to difficulty in getting CBS quality benchmark reference data. There is however a fairly large gap between Rungs 2 and 3, on account of GGAs performing considerably worse than mGGAs. We therefore recommend that only functionals from Rung 3 or higher be used for calculations where polarizabilities matter.

Several of the best performing functionals within each rung (N12, mBEEF, SCAN0, XYGJ-OS etc.) are quite recent, indicating that some measure of progress has been made in improving polarizability predictions over time (which also indirectly suggests an improvement in functional transferability and treatment of excited states). Several of the worst performers like SOGGA11, MN11-L and MN12-SX however are also recent, indicating an inconsistent improvement over time. This is in no small part due to catastrophic failures for a few species, which indicates a lack of transferability of the functional as a consequence of overfitting.

It is also quite interesting that the large performance gap between GGAs and mGGAs does not carry over to hGGAs and hmG-
GAs, which give essentially similar performance. It is possible that exact-exchange and kinetic energy density correct similar aspects in polarizability predictions, and their combination can do little better. However the hmGGA functional space is vast, and the emphasis till date has been to improve zero-field energetics alone. This has led to functionals like oB97M-V, that are highly accurate for zero-field energetics, but somewhat lacking with regards to prediction of properties like dipole moments and polarizabilities. It may therefore be possible to employ information about molecular properties to assist exploration of a larger section of functional space in order to develop hmGGAs that are simultaneously accurate for zero-field energies and properties.

We hope therefore hope that our dataset of CCSD(T)/CBS benchmark static polarizabilities will assist in development and/or testing of future functionals that predict better energy and density responses to external electric fields, and also provide a better description of dipole allowed excited states. This would ideally assist in developing more transferable functionals that would be better approximations to the exact one. Such functionals would also likely prove useful for studies that go beyond isolated systems in gas phase—such as in simulations of spectroscopy or condensed phase processes. A comparison between the supplied DFT polarizabilities and ones calculated from TDDFT could also prove useful in examining the validity of the adiabatic approximation employed in the latter, which ought to prove useful in improving DFT excited state predictions in general.

Conflicts of Interest
There are no conflicts of interest to declare.

Supporting Information
Geometries, benchmark polarizabilities and table of errors.

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